A Wavelet Algorithm for the Solution of the Double Layer Potential Equation over Polygonal Boundaries

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Abstract.
In this paper we consider a piecewise linear collocation method for the solution of the double layer potential equation corresponding to Laplace's equation over polygonal domains. We give a wavelet algorithm for the computation of the corresponding stiffness matrix and for the solution of the arising matrix equation with no more than $O(N \cdot [\log N]^8)$ arithmetic operations. The error of the resulting approximate solution is of order $O(N^{-2} \cdot [\log N]^6)$. Finally, we give some remarks on the generalization of the algorithm to the piecewise cubic collocation and present numerical tests.

Key words. potential equation, collocation, wavelet algorithm
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0 INTRODUCTION

The most popular numerical methods for the approximate solution of boundary value problems for elliptic partial differential equations are finite difference or finite element methods. However, there is a well-known alternative, the so-called boundary element method. Following this scheme, one reduces the boundary value problem for the differential equation over a given domain to a certain integral equation over the boundary of the domain. Substituting the solution of this integral equation into an integral representation formula yields the solution of the original partial differential equation. The advantages of this method in comparison to finite element or finite difference schemes consist in the facts that the approximate solution fulfills the partial differential equation exactly (Of course, the boundary conditions hold only approximately.) and that the discretization of the boundary is often simpler than that of the domain (In particular, the discretization of the boundary is easier if the domain is unbounded.). Another advantage should be the reduction of the dimension of the problem. In fact, if the partial differential equation is to be solved over a d dimensional domain, then the boundary integral equation is defined over a $d-1$ dimensional boundary manifold. Consequently, the linear systems of equations which arise after the discretization step are much smaller in the case of the boundary element method. Unfortunately, the boundary element approach leads to linear systems with dense matrices whereas the matrix of the finite element systems are sparse and admit very fast and efficient methods for the solution of the corresponding matrix equation. In other words, the boundary element algorithm is only efficient if one is able to solve the arising linear system by a comparable fast method. One should be able to solve the $N \times N$ matrix equation with no more than $O(N \cdot [\log(N)]^{\mu})$ arithmetic operation, where μ is a certain non-negative constant.

The first examples of such a fast algorithm are due to Rokhlin, Hackbusch, and Nowak [48, 32] (cf. also [30, 52]) and are based on certain Taylor or Laurent series expansions for the entries of the matrix which are far away from the main diagonal. A second algorithm is built upon the multiscale structure of the discrete operators and is due to Brandt and Lubrecht [10]. A further method using different levels of Fourier series expansions for the approximate solution together with simple parametrices for the boundary integral operator has been developed by Amosov [4] (cf. also [7, 51]). For boundary integral operators with oscillatory kernels, fast algorithms have been proposed by Rokhlin and Canning [49, 12]. The present paper is devoted to the wavelet approach which goes back to Beylkin, Coifman, and Rokhlin [8] (cf. also [2, 1, 33, 20, 21, 19, 22, 40, 24, 23]). The main idea of this method consists in choosing wavelet bases in the spaces of trial and test functions. Since the wavelet functions have small supports and are orthogonal to polynomials of small degree, a lot of the entries in the stiffness matrix corresponding to the wavelet bases are very small and can be neglected. The resulting matrix is sparse and the matrix equation can be solved quickly by a suitable iterative method. Let us remark, however, that in general the problem of computing the matrix corresponding to the wavelet bases has not been solved yet. If analytic formulas are available, then there is no problem (cf. [40]). However, a naive application of simple quadrature rules would lead to a slow algorithm with $O(N^{1+\epsilon})$ operations, where ϵ is a positive number depending on the approximation order and the moment condition of the wavelets. In particular, if the degree of the moment condition of the wavelets from the space of test functionals is equal to the order of approximation of the exact solution by functions from the trial space, then $\epsilon = 1$ and we would arrive at an $O(N^2)$ algorithm. Only for the special case of integral operators with smooth kernels, efficient algorithms including one-point quadrature rules for scaling functions with vanishing "shifted" moments or other special quadratures have been indicated by Beylkin, Coifman, and Rokhlin [8] (cf. also [24]). These quadratures (cf. Sect.4.3 and Appendix B of [8]) are not sufficient if the integral operator is a pseudo-differential operator or an operator of Calderon-Zygmund type and if the desired quadrature error is of the same size as the error of approximation by trial functions.

Now let us consider the double layer potential equation $Ax = y$ over the boundary Γ of a bounded and simply connected polygon $\Omega \subseteq \mathbb{R}^2$, where $Ax := [I + 2W]x$ with

$$2Wx(P) \ := \ 2[1/2 - d_{\Omega}(P)]x(P) + \int_{\Gamma} k(P,Q)x(Q)d_Q\Gamma, \quad P \in \Gamma \qquad (0.1)$$

$$k(P,Q) \ := \ \frac{1}{\pi}\frac{n_Q \cdot (P-Q)}{|P-Q|^2}. \tag{0.2}$$

Here $d_\Omega(P)$ denotes the normalized interior angle of Ω at the boundary point P and n_Q is the exterior unit normal of the boundary $\Gamma := \partial\Omega$ at Q. Note that this second kind integral equation is e.g. the boundary integral equation of the Dirichlet problem for Laplace's equation in Ω (cf. e.g. [37]). The kernel $k(P,Q)$ vanishes for P and Q located on the same side of Γ. It is a smooth function of P and Q if the distance between P and Q does not tend to zero. However, $k(P,Q)$ is of order $O(|P-Q|^{-1})$ if P and Q tend to a corner point but remain on different sides of Γ. In other words, the integral operator $2W$ with kernel $k(P,Q)$ has a strong singularity at the corner points of Γ. The equation $Ax = y$ is a second kind integral equation with non-compact integral operator $2W$. Nevertheless, the theorems of e.g. [19, 24] apply to the numerical solution of $Ax = y$ since the kernel $k(P,Q)$ satisfies estimates of Calderon-Zygmund type. Following this line, we get a wavelet method over uniform partitions of the boundary. The compression strategy depends on the level of the wavelets and on their location. The convergence is estimated in L^2 or in Sobolev spaces. Due to the singular behaviour of the solution x, however, the speed of convergence is slow.

In the present paper, we shall solve $Ax = y$ by a fully discretized collocation method with smoothest piecewise linear (or cubic) splines as trial functions. These trial functions will be defined using an exponential parametrization of the curve Γ. Thus the trial functions are given over a uniform grid on the parameter domain which corresponds to a grid with geometric mesh grading near the corner points over Γ. The mesh grading near corners guarantees an asymptotic L^∞- error estimate of $O(\ h^{d+1}(\log h^{-1})^\mu\ )$ for the collocation solution, where h is the mesh size, $d = 1$ ($d = 3$) is the degree of the trial functions and μ is a non-negative constant. The uniformness of the mesh in the parameter domain allows to introduce simple bases of wavelet functions. As basis functions in the trial space, we shall consider biorthogonal wavelets in the sense of [16], where the scaling function is the linear (or cubic) B-spline and the dual scaling function is an exponentially decaying function. We choose the dual scaling function such that our wavelets have two (or four) vanishing moments and that, beside this moment condition, the supports of our wavelets are minimal. Remark that small supports of the wavelet functions result in better constants for the estimates of the compression and for the estimates of the number of necessary arithmetic operations. In general, it is an open question which type of wavelets is the most convenient one. For wavelets with larger supports, the bounds for the norms of the corresponding wavelet transforms may be smaller. These bounds play a role in the convergence analysis (cf. Sects.3 and 4). For the space of test functionals, i.e., for the space spanned by the Dirac-δ distributions, we shall introduce the basis of [33, 10]. In other words, the wavelet test functionals are linear combinations of three (or five) Dirac-δ functionals. This representation is of great importance for the computation of the stiffness matrix (cf. Sect.1.4). Using these trial and test wavelets, we consider the standard form of the stiffness matrix. We shall give an easy a priori compression scheme for this matrix, i.e., we shall give a strategy for the neglect of entries depending only on the wavelet level such that the additional error caused by this neglect has the same order as the discretization error of the spline collocation without wavelets. The compressed matrix will contain no more than $O(N[\log N])$ non-zero entries. Consequently, the matrix equation can be solved in $O(N[\log N])$ operations by a suitable iteration. We recommend to take GMRES for this purpose (cf. [50, 46] and Sect.1.4). Finally, we shall give a fast algorithm to compute the compressed stiffness matrix with no more than $O(N[\log N]^8)$ operations. It will turn out that the step size of the quadrature rules applied for the computation of the entries can be chosen to be larger if the level of the test functional is high. Indeed, for this case, the entries are small and a larger relative quadrature error leads still to small absolute errors (Of course the rigorous estimates have to be shown for the global quadrature and not for each entry of the stiffness matrix.).

The plan of the paper is as follows. In Sects.1.1 and 1.2 we shall present a fully discrete collocation scheme with piecewise linear trial functions resulting in a linear system of N equations. For this collocation, we define a fast wavelet algorithm in Sects.1.3 and 1.4 which requires no more than $O(N[\log N]^8)$ arithmetic operations and a storage capacity of $O(N[\log N])$ numbers. A similar algorithm for piecewise cubic splines is described in Sect.1.5. In Sect.2 we present some numerical tests to confirm the effectiveness of the algorithm. We shall prove in Sect.3 that our discretized and compressed collocation is stable. Finally, the convergence rate $O(N^{-2}[\log N]^6)$ for the piecewise linear wavelet algorithm will be shown in Sect.4.

We remark that our method is not optimal. It has been chosen in such a manner that it admits a generalization for the case of two-dimensional polyhedral boundaries. A first step in this direction has been done in [47], where the stability of a tensor spline collocation has been proved. For an improvement of the one-dimensional method including better meshes (Remark that better meshes means meshes admitting better orders of convergence. However, the compression algorithms may be more complicated for better meshes.), superconvergence, extrapolation, multi-grid techniques, p- and h-p-methods we refer to [38, 3, 13, 36, 26, 43, 29, 53, 6, 34, 27, 39, 25, 42].

1 DESCRIPTION OF THE ALGORITHM

1.1 The collocation method

For our collocation method, we have to introduce the sets of trial functions and collocation points. To prepare this, we define a **parametrization of the polygonal boundary** Γ. Clearly, Γ is the union of straight line segments. We divide each straight line segment into two equal parts and get $\Gamma = \cup_{j=1}^{K}\Gamma_j$, where $\Gamma_j = \overline{P^jQ^j}$, the point P^j is a corner point of Γ, and Q^j the midpoint of a side of Γ. For each Γ_j, we introduce the parametrization $\Phi_j : [-\infty, 0] \longrightarrow \Gamma_j$ by $\Phi_j(s) := P^j + e^s\ \overrightarrow{P^jQ^j}$, i.e., Φ_j is the composition of the linear parametrization $[0,1] \longrightarrow \Gamma_j$ and the exponential mapping $s \mapsto e^s$.

Now let us choose a mesh parameter $\zeta > 0$, let N stand for the number of collocation points over each Γ_j $(j = 1, \dots, K)$ and define the mesh size by $h := \zeta \log N/N$. Starting from the "uniform" partition $\{t_k, k = 1, \dots, N\}$ with $t_k := -(k-1)h,\ k = 1, \dots, N-1,\ t_N := -\infty$, we get a graded mesh of **collocation points** $\{P_{(j,k)},\ j = 1, \dots, K,\ k = 1, \dots, N\}$ over Γ, where $P_{(j,k)} := \Phi_j(t_k)$ (cf. Figure 1 and compare the meshes of class $\mathcal{M}$ in Sect.5.16 of [42]). Note that this mesh is geometrically graded towards the corner points $P^j = P_{j,N}$, i.e.,

$$|P_{(j,k+1)} - P_{(j,k)}| = e^{-h}|P_{(j,k)} - P_{(j,k-1)}|,\ k = 1, \dots, N-2. \tag{1.1}$$

The grading factor e^{-h}, however, tends to one for $N \longrightarrow \infty$. The mesh size $\sup_{j,k} |P_{(j,k)} - P_{(j,k-1)}|$ is of order $O(1-e^{-h}) = O(h)$ and the subinterval adjacent to the corner $P^j =$

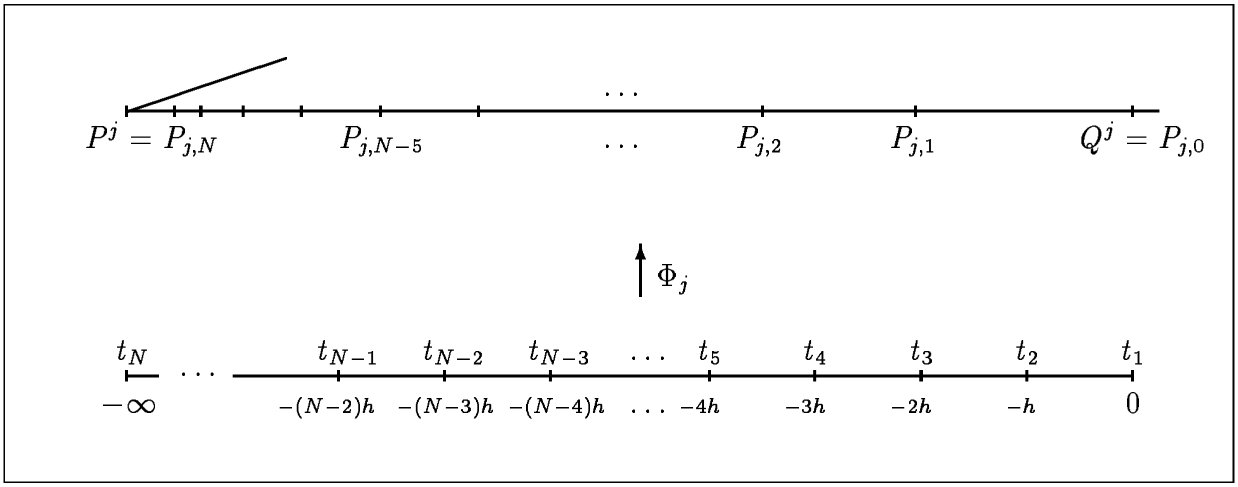

Figure 1: Grid points on $(-\infty, 0]$ and Γ.

$P_{j,N}$ is of length $O(e^{-h[N-2]}) = O(N^{-\zeta})$.

For the definition of **trial functions**, we first introduce a piecewise linear spline basis over the mesh $\{-(k-1)h,\ k = 1, \dots N-1\}$. Let φ stand for the linear B-spline

$$\varphi : I\!R \longrightarrow I\!R,\ \varphi(t) := \begin{cases} 1+t & \text{if } -1 < t \leq 0 \\ 1-t & \text{if } \quad 0 < t \leq 1 \\ 0 & \text{else.} \end{cases} \tag{1.2}$$

We define $\varphi_k : [-\infty, 0] \longrightarrow I\!R$ by $\varphi_k(s) := \varphi(s/h + k - 1),\ k = 1, \dots, N-1$ and set $\varphi_N(s) := 1 - \sum_{k=1}^{N-1} \varphi_k(s)$, i.e., $\varphi_N(s) := \varphi(s/h + N - 1)$ if $s \geq -(N-1)h$ and $\varphi_N(s) := 1$ if $s < -(N-1)h$. Using our parametrization we introduce the final basis functions $\varphi_{(j,k)} : \Gamma_j \longrightarrow I\!R,\ j = 1, \dots, K,\ k = 1, \dots, N$ by

$$\varphi_{(j,k)}(\Phi_m(s)) := \begin{cases} \varphi_k(s) & \text{if } j = m \\ 0 & \text{else }, \end{cases} \qquad m = 1, \dots, K. \tag{1.3}$$

Let us note that the $\varphi_{(j,k)}$ span the whole space of parameterized linear splines over the intervals $[\Phi_j(-(N-1)h), \Phi_j(0)]$. Over $[\Phi_j(-\infty), \Phi_j(-(N-1)h)]$ the span contains only the constant functions. However, the last subinterval is of size $O(N^{-\zeta})$ and, if $\zeta \geq 2$, then any smooth function can be approximated by a function from the span of $\varphi_{(j,k)}$ with order $O(h^2)$. In order to simplify the notation, we introduce the index set $I := \{(j,k) :\ j = 1, \dots, K,\ k = 1, \dots, N\}$ and denote its elements by ι, κ, i.e., for $\iota, \kappa \in I$ we set $\iota = (j_\iota, k_\iota),\ \kappa = (j_\kappa, k_\kappa)$.

Now the **collocation method** for the numerical solution of $Ax = y$ consists in seeking an approximate solution $x_N = \sum_{\iota \in I} \xi_\iota \varphi_\iota$ with real coefficients ξ_ι satisfying

$$Ax_N(P_\kappa) = y(P_\kappa),\ \kappa \in I. \tag{1.4}$$

Note that each end point P^j, Q^j of the straight line segment Γ_j appears twice in the set of collocation points. We shall distinguish these points formally and, for a function f piecewise continuous over Γ and continuous over each Γ_j, we set $f(P_{(j,k)}) = \lim_{\Gamma_j \ni Q \to P_{(j,k)}} f(Q)$. With respect to the coefficients ξ_ι the collocation equations (1.4) form a linear system of equation. We denote its matrix $((A\varphi_\iota)(P_\kappa))_{\kappa,\iota \in I}$ by $A_N =$

$(a_{\kappa,\iota})_{\kappa,\iota\in I}$. This matrix is called stiffness matrix of the collocation. It is well known that the collocation (1.4) fits in the frame of Galerkin-Petrov methods. Indeed collocation seeks an approximate solution x_N in the space $span\{\varphi_\iota,\ \iota\in I\}$ such that $\vartheta(Ax_N - y) = 0$ for any functional ϑ from the space of test functionals $span\{\delta_{P_\kappa},\ \kappa\in I\}$.

1.2 The discretized collocation

Method (1.4) represents only a semi-discretization since the computation of the entries $a_{\kappa,\iota}$ of the stiffness matrix A_N requires an integration. In our discretized collocation method we shall replace this integration by **simple quadrature rules**. Thus let us introduce quadrature rules and start with rules over $[-\infty, 0]$. Taking into account that the trial functions $\varphi_k,\ k = 1,\dots,N$ are constant over $[-\infty, -h(N-1)]$, we take the rule

$$\begin{aligned} \int_{-\infty}^{0} f(e^s)e^s ds &= \int_0^{e^{-(N-1)h}} f(x)dx + \int_{-(N-1)h}^{0} f(e^s)e^s ds \\ &\sim Q_1(f; 0, e^{-(N-1)h}) + Q_2(f; -(N-1)h, 0) \\ &=: \sum_{\lambda=1}^{\tilde N} f(\sigma_\lambda)\tilde\omega_\lambda. \end{aligned} \tag{1.5}$$

Here $Q_2(f; -(N-1)h, 0)$ denotes the composite trapezoidal rule corresponding to the partition $\{-kh :\ k = 0,\dots,N-1\}$ of $[-(N-1)h, 0]$ and applied to the function $[-(N-1)h, 0] \ni s \mapsto f(e^s)e^s$. The symbol $Q_1(f; 0, e^{-(N-1)h})$ stands for the composite trapezoidal rule corresponding to the partition $\{-ke^{-(N-1)h}/i_* :\ k = 0,\dots,i_*\}$ of $[0, e^{-(N-1)h}]$ and applied to the function $[0, e^{-(N-1)h}] \ni x \mapsto f(x)$. For the discretized collocation without wavelet algorithm, the number i_* is an a priori fixed positive integer which is independent of h and N. Using the parametrization Φ_j, we arrive at the quadrature rule

$$\begin{aligned} \int_\Gamma f(Q) d_Q\Gamma &= \sum_{j=1}^{K} \int_{-\infty}^{0} f(\Phi^j(s))e^s ds |\ \overrightarrow{P^jQ^j}\ | \\ &\sim \sum_{\mu\in J} f(Q_\mu)\omega_\mu\ , \\ J &:= \{\mu = (j_\mu, \lambda_\mu) :\ j_\mu = 1,\dots,K, \lambda_\mu = 1,\dots,\tilde N\}, \\ Q_\mu &:= \Phi_{j_\mu}(\sigma_{\lambda_\mu}),\ \omega_\mu := |\ \overrightarrow{P^{j_\mu}Q^{j_\mu}}\ |\tilde\omega_{\lambda_\mu}. \end{aligned} \tag{1.6}$$

Preparing the application of our quadrature rule to the integral in $a_{\kappa,\iota}$, we perform a step which is called **singularity subtraction** or regularization or modified quadrature method. Using $W1 = 1/2$ (cf. [37]), we write

$$(A\varphi_\iota)(P_\kappa) = \varphi_\iota(P_\kappa) + \varphi_\iota(P_{\kappa_1}) + \int_\Gamma k(P_\kappa, Q)[\varphi_\iota(Q) - \varphi_\iota(P_{\kappa_1})]d_Q\Gamma. \tag{1.7}$$

Here $\kappa_1 := \kappa$ if P_κ is not a corner point. If P_κ is a corner point with $\{P_\kappa\} = \Gamma_{j_\kappa} \cap \Gamma_j$, then $\kappa_1 := (j, N)$. I.e., for corner points P_κ, κ_1 is just the index of I different from κ such that $P_\kappa = P_{\kappa_1}$. Applying (1.6) with mesh size h to (1.7) yields

$$a_{\kappa,\iota} \sim a'_{\kappa,\iota} \;=\; \varphi_\iota(P_\kappa) + [1-\Sigma_\kappa]\varphi_\iota(P_{\kappa_1}) + \sum_{\mu\in J} k(P_\kappa,Q_\mu)\omega_\mu\varphi_\iota(Q_\mu), \tag{1.8}$$
$$\Sigma_\kappa \;:=\; \sum_{\mu\in J} k(P_\kappa,Q_\mu)\omega_\mu.$$

Thus the discretized collocation is nothing else than the method (1.4), where the matrix $(a_{\kappa,\iota})_{\kappa,\iota\in I}$ of the system of equations is replaced by $A'_N := (a'_{\kappa,\iota})_{\kappa,\iota\in I}$. In order to motivate the singularity subtraction let us mention that the replacement of $a_{\kappa,\iota}$ by $a'_{\kappa,\iota}$ corresponds to the approximation

$$\begin{aligned}(Ax_N)(P_\kappa) &= x_N(P_\kappa) + x_N(P_{\kappa_1}) + \int_\Gamma k(P_\kappa,Q)[x_N(Q)-x_N(P_{\kappa_1})]d_Q\Gamma \\ &\sim x_N(P_\kappa) + x_N(P_{\kappa_1}) + \sum_{\mu\in J} k(P_\kappa,Q_\mu)[x_N(Q_\mu)-x_N(P_{\kappa_1})]\omega_\mu.\end{aligned} \tag{1.9}$$

No singularity subtraction results in

$$\begin{aligned}(Ax_N)(P_\kappa) &= x_N(P_\kappa) + 2[\frac{1}{2} - d_\Omega(P_{\kappa_1})]x_N(P_{\kappa_1}) + \int_\Gamma k(P_\kappa,Q)x_N(Q)d_Q\Gamma \\ &\sim x_N(P_\kappa) + 2[\frac{1}{2} - d_\Omega(P_{\kappa_1})]x_N(P_{\kappa_1}) + \sum_{\mu\in J} k(P_\kappa,Q_\mu)x_N(Q_\mu)\omega_\mu.\end{aligned} \tag{1.10}$$

Since the kernel function k has a certain strong singularity at the corner points, the quadratures for $\int_\Gamma k(P_\kappa,Q)x_N(Q)d_Q\Gamma$ do not converge uniformly with respect to κ. The expression $k(P_\kappa,Q)[x_N(Q)-x_N(P_{\kappa_1})]$ has a milder singularity as $k(P_\kappa,Q)x_N(Q)$ if x_N is smooth. Consequently, the quadratures of $\int_\Gamma k(P_\kappa,Q)[x_N(Q)-x_N(P_{\kappa_1})]d_Q\Gamma$ converge uniformly. In other words, the discretized collocation method without subtraction technique is not convergent in L^∞ whereas the discretized collocation method with subtraction technique converges with the same order as the collocation method.

1.3 The wavelet bases

Next we introduce new bases in the space of trial functions and in the space of test functionals, respectively. Let us start with the bases over $[-\infty,0]$ and with the **wavelet basis in the space of test functionals over the half axis**. We consider a fixed N of the form $N = 7\cdot 2^{lev}+1$ and the corresponding $h := \zeta \log N/N$. Over the real axis $\mathbb{R}$ we have a hierarchy of grids $\{-kh2^{lev-l},\ k\in\mathbb{Z}\}$, $l=0,\ldots,lev$ and the corresponding partition $\{-kh,\ k\in\mathbb{Z}\} = \{-kh2^{lev},\ k\in\mathbb{Z}\}\cup\cup_{l=1,\ldots,lev}\{-(2k+1)h2^{lev-l},\ k\in\mathbb{Z}\}$. Analogously, for the grid points $\{t_k,\ k=1,\ldots,N\}$, we get the partition $\cup_{l=0,\ldots,lev}\{t^l_k,\ k=1,\ldots,N^T_l\}$, where

$$\begin{aligned} t^0_k &:= -(k-1)h2^{lev},\ \ k=1,\ldots,N^T_0-1,\ \ t^0_{N^T_0} := -\infty,\ \ N^T_0 := 8 \\ t^l_k &:= -(2k-1)h2^{lev-l},\ \ k=1,\ldots,N^T_l,\ \ l=1,\ldots,lev,\ \ N^T_l := 7\cdot 2^{l-1}.\end{aligned} \tag{1.11}$$

For $l=0$, we set $\vartheta^0_k := \delta_{t^0_k}$, $k=1,\ldots,N^T_0$, i.e., $\vartheta^0_k(f) := f(t^0_k)$. For $l>0$, we choose ϑ^l_k to be the linear combination

$$\vartheta_k^l := \delta_{t_k^l} - \sum_{j=1}^{2} \alpha_{k,j}^l \delta_{t_{k,j}^l} \tag{1.12}$$

of three Dirac-δ functionals, where $t_{k,1}^l$ and $t_{k,2}^l$ are the two grid points of the coarser levels $\cup_{m=0,\dots,l-1}\{t_k^m,\ k=0,\dots,N_m^T\}$ nearest to t_k^l. In other words,

$$t_{k,1}^l := \begin{cases} -h2^{lev-(l-1)}(k-1) & \text{if } k < N_l^T \\ -h2^{lev-(l-1)}(k-2) & \text{if } k = N_l^T, \end{cases} \quad t_{k,2}^l := \begin{cases} -h2^{lev-(l-1)}k & \text{if } k < N_l^T \\ -h2^{lev-(l-1)}(k-1) & \text{if } k = N_l^T. \end{cases} \tag{1.13}$$

The coefficients $\alpha_{k,j}^l$ are chosen such that the wavelet functional ϑ_k^l vanishes at all linear functions, i.e., we define

$$\alpha_{k,1}^l := \begin{cases} 1/2 & \text{if } k < N_l^T \\ -1/2 & \text{if } k = N_l^T, \end{cases} \qquad \alpha_{k,2}^l := \begin{cases} 1/2 & \text{if } k < N_l^T \\ 3/2 & \text{if } k = N_l^T. \end{cases} \tag{1.14}$$

It is not hard to see that $span\{\vartheta_k^l :\ k=1,\dots,N_l^T,\ l=0,\dots,lev\} = span\{\delta_{t_k},\ k = 1,\dots,N\}$. This wavelet basis is a special case of the wavelets in [33].

Now we turn to the **wavelet basis for the space of trial functions**. Let us start with the wavelets over the **real axis**. Analogously to [55, 16] we introduce

$$\psi(s) := \frac{1}{2}\sum_{j=0}^{2}\binom{2}{j}(-1)^j\varphi(s-j+1) \tag{1.15}$$

and obtain that $span\{\varphi(s-k),\ k\in\mathbb{Z}\}$ is the direct sum of $span\{\varphi(s/2-k),\ k\in\mathbb{Z}\}$ and $span\{\psi(s-(2k-1)),\ k\in\mathbb{Z}\}$. Hence a wavelet basis over $\mathbb{R}$ can be given by

$$\begin{aligned} \tilde\psi_k^0(s) &:= \varphi(s/(h2^{lev})-k),\ \ k\in\mathbb{Z}, \\ \tilde\psi_k^l(s) &:= \psi(s/(h2^{lev-l})-(2k-1)),\ \ k\in\mathbb{Z}, l=1,\dots,lev. \end{aligned} \tag{1.16}$$

Note that all $\tilde\psi_k^l$ with $l>0$ are orthogonal to linear functions, i.e., they have two vanishing moments $\int\tilde\psi_k^l(s)ds=0$, $\int\tilde\psi_k^l(s)sds=0$. In the class of all wavelet bases with this orthogonality property our wavelets have minimal support.

Similarly to the wavelets over the interval (cf. [5, 15, 17]), the **wavelet basis of the trial space over the half axis** will consist of interior wavelets and boundary wavelets. The interior wavelets are just those wavelets on the real axis the support of which is contained in $(-(N-1)h, 0)$. The boundary wavelets are certain modifications of those wavelets defined on the axis which do not vanish at 0 or at $-(N-1)h$. We shall choose them in such a way that the transformation from the basis of scaling functions $\{\varphi_k,\ k=1,\dots,N\}$ into the new basis of wavelets is bounded. We do not care about the moment condition for boundary wavelets. To introduce the basis we observe that all piecewise linear functions over $[-\infty,0]$ can be extended to an even function of the space $span\{\Theta_k(s) := \varphi(s/h-k)+\varphi(s/h+k),\ k=0,1,\dots\}$ over $\mathbb{R}$ by reflection. Taking the wavelet basis $\{\tilde\Theta_k^0(s) := \varphi(s/(h2^{lev})-k)+\varphi(s/(h2^{lev})+k),\ k=0,1,\dots\}\cup\{\tilde\Theta_k^l(s) := \psi(s/(h2^{lev-l})-(2k-1))+\psi(s/(h2^{lev-l})+(2k-1)),\ k=1,2,\dots,\ l=1,\dots,lev\}$ of this spline space and restricting it to the half axis $[-\infty,0]$, we arrive at a wavelet basis on $[-\infty,0]$ with bounded wavelet transform. Together with a corresponding modification

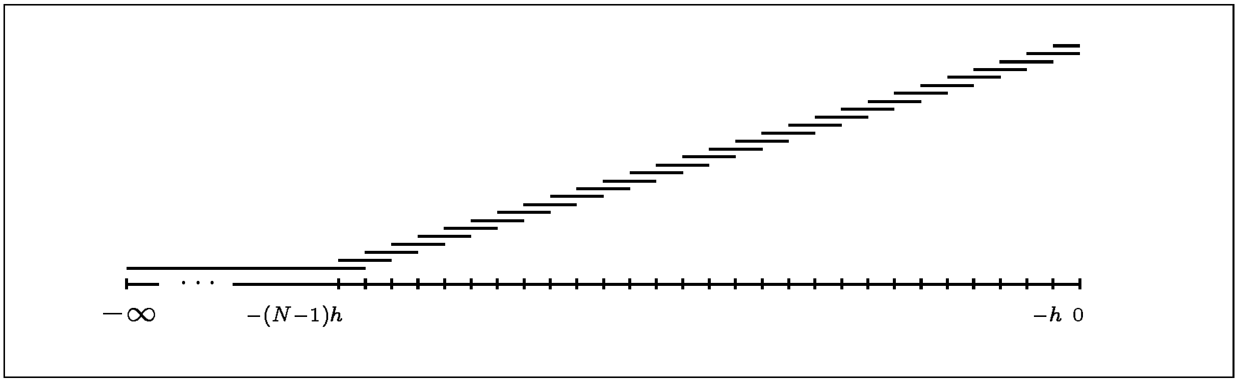

Figure 2: Supports of the functions φ_k over $[-\infty, 0]$.

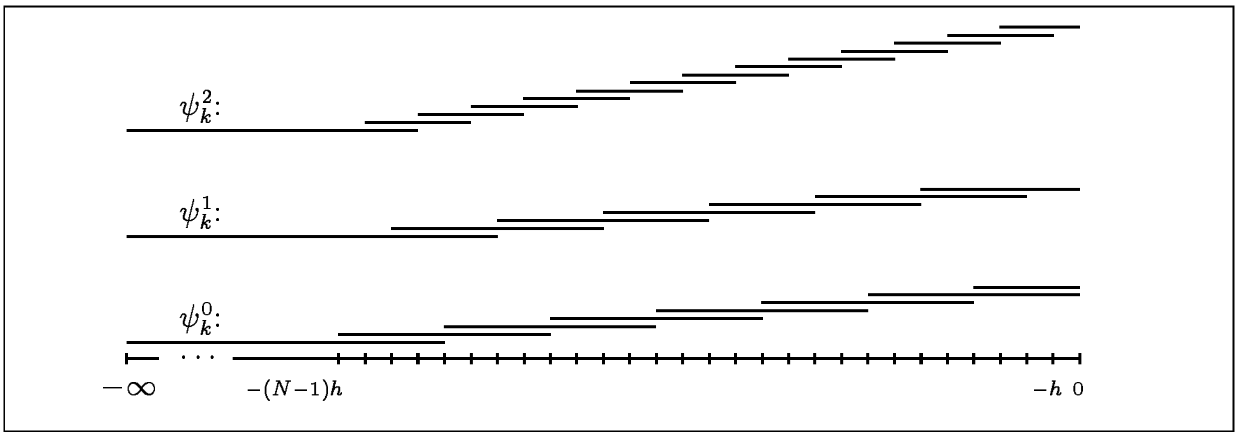

Figure 3: Supports of the functions ψ_k^l over $[-\infty, 0]$.

over $[-\infty, -(N-1)h]$, we get the following definition (cf. Figures 2 and 3 for the supports of the functions):

$$
\begin{aligned}
\psi_k^0(s) &:= \varphi(s/(h2^{lev}) + k - 1),\ k = 1, \ldots, N_0^A - 1,\ N_0^A := 8, \qquad (1.17)\\
\psi_{N_0^A}^0(s) &:= \begin{cases} \varphi(s/(h2^{lev}) + N_0^A - 1) & \text{if } s \geq -h(N-1) \\ 1 & \text{if } s < -h(N-1)\,, \end{cases}\\
\psi_1^l(s) &:= \psi(s/(h2^{lev-l}) - 1) + \psi(s/(h2^{lev-l}) + 1),\\
\psi_k^l(s) &:= \psi(s/(h2^{lev-l}) + (2k-1)),\quad k = 2, \ldots, N_l^A - 1,\quad N_l^A := 7 \cdot 2^{l-1},\\
\psi_{N_l^A}^l(s) &:= \begin{cases} \psi(s/(h2^{lev-l}) + (2N_l^A - 1)) + & \text{if } s \geq -h(N-1) \\ \quad \psi(s/(h2^{lev-l}) + (2N_l^A + 1)) & \\ 1 & \text{if } s < -h(N-1)\ , \end{cases}\\
& \qquad l = 1, \ldots, lev.
\end{aligned}
$$

Clearly, the ψ_k^l with $k = 2, \ldots, N_l^A - 1,\ l = 1, \ldots, lev$ are interior wavelets and ψ_1^l as well as $\psi_{N_l^A}^l$ are boundary wavelets.

After the introduction of the wavelet bases over $[-\infty, 0]$, we get the final **wavelet bases over the curve** Γ using our parametrizations. We define the index sets $I^A := \{\iota = (j_\iota, l_\iota, k_\iota) :\ j_\iota = 1, \ldots, K,\ l_\iota = 0, \ldots, lev,\ k_\iota = 1, \ldots, N_l^A\}$ and $I^T := \{\kappa = (j_\kappa, l_\kappa, k_\kappa) :\ j_\kappa = 1, \ldots, K,\ l_\kappa = 0, \ldots, lev,\ k_\kappa = 1, \ldots, N_l^T\}$ (Note that $I^A = I^T$ for the case of linear splines.). For $\iota \in I^A$, we define the wavelet function ψ_ι by

$$\psi_{(j_\iota,l_\iota,k_\iota)}(\Phi_m(s)) := \begin{cases} \psi^{l_\iota}_{k_\iota}(s) & \text{if } j_\iota = m \\ 0 & \text{else.} \end{cases} \tag{1.18}$$

Obviously, $span\{\varphi_\iota,\ \iota \in I\} = span\{\psi_\iota,\ \iota \in I^A\}$. To define the basis in the space of test functionals, we take $\kappa \in I^T$ and set

$$\hat{P}_{(j_\kappa,l_\kappa,k_\kappa)}(f) := \vartheta^{l_\kappa}_{k_\kappa}(f \circ \Phi_{j_\kappa}). \tag{1.19}$$

For simplicity of notation, let us look at the functionals $\hat{P}_\kappa$ as if they were Dirac-δ distributions at a point $\hat{P}_\kappa$ and write $f(\hat{P}_\kappa)$ instead of $\hat{P}_\kappa(f)$.

Using the just defined wavelet bases, we arrive at a **transformed stiffness matrix** $B_N := (A\psi_\iota(\hat{P}_\kappa))_{\kappa\in I^T,\iota\in I^A}$. It turns out that the entry $A\psi_\iota(\hat{P}_\kappa)$ is small and negligible if the levels l_ι, l_κ of the wavelets are large and if ψ_ι is not a boundary wavelet. Thus we replace B_N by the **compressed matrix** $B^c_N := (b^c_{\kappa,\iota})_{\kappa\in I^T,\iota\in I^A}$, where $b^c_{\kappa,\iota} := A\psi_\iota(\hat{P}_\kappa)$ if $\psi_\iota \neq 0$ over $supp\,\hat{P}_\kappa$ or if ψ_ι is a boundary wavelet or if $l_\iota \leq lev - l_\kappa$ and $b^c_{\kappa,\iota} := 0$ else (For a compression with a larger number of neglected entries we refer to Remark 4.4.). This compressed matrix is a small perturbation of B_N and contains no more than $O(N[\log N])$ (cf. Sect.1.4) non-vanishing entries. The matrix equation with matrix B^c_N can be solved with at most $O(N[\log N])$ arithmetic operations.

1.4 The wavelet algorithm

Our next concern is to give an algorithm for the computation of a discretized version of the matrix B^c_N. To this end let us proceed analogously to Sect.1.2. However, before we describe the algorithm for the computation of the entries $a_{\kappa,\iota}$, let us introduce a **quadrature rule similar to (1.6) but with coarser mesh size**. Clearly, $Q_2(f;-(N-1)h,0)$ in (1.5) is the trapezoidal rule over a partition with mesh size h. Therefore, we call (1.6) including this $Q_2(f;-(N-1)h,0)$ the rule (1.6) with mesh size h. Now suppose $N = 7\cdot 2^{lev}+1$, $l \leq lev$, and consider the mesh size $h_{qu} := 2^l \cdot h$ for the quadrature. We replace $Q_2(f;-(N-1)h,0)$ in (1.5) by the composite trapezoidal rule applied to the function $[-(N-1)h,0] \ni s \mapsto f(e^s)e^s$ over the partition $Part$ of $[-(N-1)h,0]$, where $Part$ is the union of $\{-kh_{qu},\ k=0,\ldots,2^{-l}\cdot(N-1)\}$ with

$$\bigcup_{m=0,\ldots,l-1}\Big\{-k(h\cdot 2^m):\ k=0,1,2,3\Big\}\cup \tag{1.20}$$
$$\bigcup_{m=0,\ldots,l-1}\Big\{-k(h\cdot 2^m):\ k=2^{-m}\cdot(N-1)-3,\ldots,2^{-m}\cdot(N-1)\Big\}\quad.$$

Furthermore, we shall choose $i_* := lev^3$ in the the definition of $Q_1(f;0,e^{-(N-1)h})$. These two modifications result in a new quadrature rule (1.6) which we call (1.6) with mesh size h_{qu}. Note that the partition $Part$ in this quadrature rule is chosen such that the quadrature rule is exact for all trial wavelet functions which remain after the compression step (cf. the compressed matrix at the end of Sect.1.3 and the set $I^A(\hat{P}_\kappa)$ in the following algorithm). The uniform partition $\{-kh_{qu},\ k=0,\ldots,2^{-l}\cdot(N-1)\}$ guarantees the exactness of the quadrature to the integrals of the wavelets $\psi_{(j_\iota,l_\iota,k_\iota)}$ with level l_ι less or equal to $lev-l$. The node points from (1.20) guarantee the exactness of the quadrature for the integrals of the boundary wavelets $\psi_{(j_\iota,l_\iota,1)}$ and $\psi_{(j_\iota,l_\iota,N^A_{l_\iota})}$.

By definition (cf. (1.12)) each functional $\hat{P}_\kappa$ is the linear combination of at most three Dirac-δ functionals, i.e., there exist $\alpha_1,\ \alpha_2,\ \alpha_3 \in I\!R$ and $P_{\kappa,1},\ P_{\kappa,2},\ P_{\kappa,3} \in \Gamma$ such that $f(\hat{P}_\kappa) = \sum_{i=1}^3 \alpha_i f(P_{\kappa,i})$. Hence, for the **singularity subtraction**, we get

$$(Ax_N)(\hat{P}_\kappa) = \sum_{i=1}^{3} \alpha_i \left\{ x_N(P_{\kappa,i}) + x_N(P_{\kappa,i}^+) + \int_\Gamma k(P_{\kappa,i},Q)[x_N(Q) - x_N(P_{\kappa,i}^+)]d_Q\Gamma \right\}, \tag{1.21}$$

where $P_{\kappa,i}^+ := P_{\kappa,i}$ if $P_{\kappa,i}$ is not a corner point of Γ. If $P_{\kappa,i}$ is a corner point and $x_N(P_{\kappa,i})$ is the limit of x_N from the side Γ_{j_κ} of Γ, then $P_{\kappa,i}^+$ stands for the same corner point $P_{\kappa,i}$ but $x_N(P_{\kappa,i}^+)$ is the limit from the side $\Gamma \setminus \Gamma_{j_\kappa}$. Following the compression strategy of the matrix B_N^c, we replace $x_N = \sum_{\iota \in I^A} \xi_\iota \psi_\iota$ by $x_N^c = \sum_{\iota \in I^A(\hat{P}_\kappa)} \xi_\iota \psi_\iota$, where $I^A(\hat{P}_\kappa)$ is the set of all $\iota \in I^A$ such that $\psi_\iota(P_{\kappa,i}) \neq 0,\ i = 1,2,3$ or that ψ_ι is a boundary wavelet or that $l_\iota \le lev - l_\kappa$. Since $x_N^c(P_{\kappa,i}) = x_N(P_{\kappa,i})$, we get

$$(Ax_N)(\hat{P}_\kappa) \sim \sum_{i=1}^{3} \alpha_i \left\{ x_N(P_{\kappa,i}) + x_N(P_{\kappa,i}^+) + \int_\Gamma k(P_{\kappa,i},Q)[x_N^c(Q) - x_N^c(P_{\kappa,i}^+)]d_Q\Gamma \right\}. \tag{1.22}$$

Let us choose $h_{qu} = \min(h \cdot 2^{l_\kappa}, h \cdot 2^{lev - lev_0})$ with $lev_0 := 7[\log lev / \log 2]$ and apply (1.6) with mesh size h_{qu} to (1.22). We obtain

$$\begin{aligned} (Ax_N)(\hat{P}_\kappa) \quad &\sim \quad \sum_{i=1}^{3} \alpha_i \left\{ x_N(P_{\kappa,i}) + [1 - \Sigma_{\kappa,i}] x_N(P_{\kappa,i}^+) + \sum_{\mu \in J} k(P_{\kappa,i}, Q_\mu) x_N^c(Q_\mu)\omega_\mu \right\}. \\ &= \quad x_N(\hat{P}_\kappa) + \sum_{i=1}^{3} \alpha_i [1 - \Sigma_{\kappa,i}] x_N(P_{\kappa,i}^+) + \sum_{\mu \in J} k(\hat{P}_\kappa, Q_\mu) x_N^c(Q_\mu)\omega_\mu\ , \qquad (1.23) \\ \Sigma_{\kappa,i} \quad &:= \quad \sum_{\mu \in J} k(P_{\kappa,i}, Q_\mu)\omega_\mu . \end{aligned}$$

For the approximate value $b'_{\kappa,\iota}$ of the entry $b^c_{\kappa,\iota}$ of B_N^c, this leads to

$$b'_{\kappa,\iota} := \begin{cases} \psi_\iota(\hat{P}_\kappa) + \sum_{i=1}^3 \alpha_i [1 - \Sigma_{\kappa,i}] \psi_\iota(P_{\kappa,i}^+) + \\ \qquad \sum_{\mu \in J} k(\hat{P}_\kappa, Q_\mu) \psi_\iota(Q_\mu)\omega_\mu & \text{if } \iota \in I^A(\hat{P}_\kappa) \\ 0 & \text{else}. \end{cases} \tag{1.24}$$

Simply applying (1.24), we arrive at the following **algorithm for the computation of the transformed, compressed, and discretized stiffness matrix** $B'_N := (b'_{\kappa,\iota})_{\kappa \in I^T, \iota \in I^A}$.

For all $\kappa \in I^T$ do (i.e., compute successively all the rows of B'_N):

- Before summing up all the terms of $b'_{\kappa,\iota}$ and $\Sigma_{\kappa,i}$ indicated in (1.24), set $b'_{\kappa,\iota} = 0,\ \Sigma_{\kappa,i} = 0$ for any $\iota \in I^A$ and $i = 1,2,3$.

- In accordance with (1.12) and (1.19), compute the α_i, $P_{\kappa,i}$ and $P_{\kappa,i}^+$ with $i = 1,2,3$ for the test functional $\hat{P}_\kappa$.

- Set $h_{qu} = min(h \cdot 2^{l_\kappa}, h \cdot 2^{lev - lev_0})$ and compute the nodes Q_μ and the weights ω_μ of the quadrature rule (1.6) with mesh width h_{qu} (cf. the beginning of this section).

- For all $\mu \in J$ do:
  - Compute the values of the kernel function $k(P_{\kappa,i}, Q_\mu)$, $i = 1, 2, 3$.
  - Add $k(P_{\kappa,i}, Q_\mu)\omega_\mu$ to $\Sigma_{\kappa,i}$, $i = 1, 2, 3$.
  - Determine the index set $I^A(\mu)$ of all $\iota \in I^A(\hat{P}_\kappa)$ such that $\psi_\iota(Q_\mu) \neq 0$.
  - For any $\iota \in I^A(\mu)$ and for $i = 1, 2, 3$, add $\alpha_i k(P_{\kappa,i}, Q_\mu)\omega_\mu\psi_\iota(Q_\mu)$ to $b'_{\kappa,\iota}$.
- Determine the index set $J^A(\kappa)$ of all $\iota \in I^A$ such that $\psi_\iota(P_{\kappa,i}) \neq 0$ or $\psi_\iota(P^+_{\kappa,i}) \neq 0$, $i = 1, 2, 3$.
- For any $\iota \in J^A(\kappa)$, add $\alpha_i\psi_\iota(P_{\kappa,i})$ to $b'_{\kappa,\iota}$, $i = 1, 2, 3$.
- For any $\iota \in J^A(\kappa)$, add $\alpha_i[1 - \Sigma_{\kappa,i}]\psi_\iota(P^+_{\kappa,i})$ to $b'_{\kappa,\iota}$, $i = 1, 2, 3$.

Let us count the **number of arithmetic operations** of this algorithm. We observe (cf. Figure 3) that the number of wavelet functions not vanishing at a fixed point of Γ is less or equal to $2\,lev$. Hence the index sets $I^A(\mu)$ and $J^A(\kappa)$ contain no more than $O(lev)$ indices. The number of arithmetic operations for the computation of the κ-th row of B'_N is less than $O(lev)$ times the number of quadrature nodes, i.e., less than $O(lev\cdot[lev^3+2^{lev-l_\kappa}]) = O(lev\cdot 2^{lev-l_\kappa})$ if $l_\kappa < lev-lev_0$ and $O(lev\cdot[lev^3+2^{lev_0}]) = O(lev^8)$ else. For the computation of the whole matrix we need a number of operations of order

$$\begin{aligned} O\left(\sum_{l_\kappa=lev-lev_0}^{lev} 2^{l_\kappa} lev^8 + \sum_{l_\kappa=0}^{lev-lev_0-1} 2^{l_\kappa} lev \cdot 2^{lev-l_\kappa}\right) &= O(lev^8 \cdot 2^{lev}) \qquad (1.25)\\ &= O(N[\log N]^8). \end{aligned}$$

Let us count the **number of non-zero entries** in B'_N. The number in one row is just the cardinality of $I^A(\hat{P}_\kappa)$. There exist no more than $O(lev)$ indices ι such that $\psi_\iota(P_{\kappa,i}) \neq 0$ or $\psi_\iota(P^+_{\kappa,i}) \neq 0$, $i = 1, 2, 3$ or that ψ_ι is a boundary wavelet. The number of indices ι with $l_\iota \leq lev - l_\kappa$ is $O(2^{lev-l_\kappa})$. Hence the κ-th row of B'_N contains at most $O(2^{lev-l_\kappa} + lev)$ entries different from 0. Consequently, the number of non-zero entries of the whole matrix B'_N is less than

$$O\left(\sum_{l_\kappa=0}^{lev} 2^{l_\kappa}\left[2^{lev-l_\kappa} + lev\right]\right) = O(lev \cdot 2^{lev}) = O(N[\log N]). \qquad (1.26)$$

In other words the storage of the matrix B'_N requires a storage capacity of $O(N[\log N])$ numbers. The computation of B'_N requires $O(N[\log N]^8)$ operations and the multiplication of B'_N by a vector $O(N[\log N])$.

Now the algorithm for the **computation of the approximate solution** x_N of equation $Ax = y$ via discretized collocation and wavelet transform looks as follows. We determine the right-hand side $y_N := (y(P_\kappa))_{\kappa\in I}$ of the collocation system (1.4) and solve $A_N x_N = y_N$ by an iterative method (e.g. by GMRES). If we choose the initial vector for our iteration to be the solution of a collocation over a coarser grid, then we need only a finite number of iteration steps to solve the collocation system up to the discretization error. The main part of this process is the matrix multiplication of the iteration vectors

z_N by A_N. This multiplication will be realized in three steps. All the three steps require no more than $O(N[\log N]^8)$ operations. Thus the whole algorithm for the computation of x_N requires no more than $O(N[\log N]^8)$ operations and a storage capacity of $O(N[\log N])$ numbers.

Now let us describe the **three steps of the multiplication** of A_N by a vector z_N. To this end, let us identify the vector $z_N = \{\xi_\iota\}_{\iota\in I}$ with the function $z_N = \sum_{\iota\in I}\xi_\iota\varphi_\iota$ of the trial space, i.e., we identify the function z_N with the vector $\{\xi_\iota = z_N(P_\iota)\}_{\iota\in I}$. Analogously, we identify the vector $[A_N z_N]$ with the function $[A_N z_N] := \sum_{\iota\in I}[A_N z_N]_\iota\varphi_\iota$ such that the κ-th entry of vector $[A_N z_N]$ is equal to the value $[A_N z_N](P_\kappa)$ of the function $[A_N z_N]$. With this notation the function z_N is given by the vector $\{\xi_\iota\}_{\iota\in I}$ of its coefficients and to compute the multiplied vector $A_N z_N$ means to compute the vector $\{[A_N z_N](P_\kappa)\}_{\kappa\in I}$. In the first step we apply the wavelet transform, i.e., we compute the coefficients η_ι of the representation $z_N = \sum_{\iota\in I^A}\eta_\iota\psi_\iota$. This step can be realized with the aid of a pyramid type scheme and is well known to require no more than $O(N)$ operations (cf. e.g. [18, 14]). We shall describe this pyramid type scheme at the end of this section. In the second step of the multiplication procedure we multiply $\{\eta_\iota\}_{\iota\in I^A}$ by the sparse matrix B_N'. Since B_N' is a small perturbation of $B_N := (A\psi_\iota(\hat{P}_\kappa))_{\kappa\in I^T,\iota\in I^A}$, we arrive at an approximation for $\{[A_N z_N](\hat{P}_\kappa)\}_{\kappa\in I^T}$. It remains to apply the inverse wavelet transform which computes, for the function $f = [A_N z_N]$, the vector $\{f(P_\kappa)\}_{\kappa\in I}$ from $\{f(\hat{P}_\kappa)\}_{\kappa\in I^T}$. This third step can also be realized with the aid of a fast pyramid type scheme which we shall present next.

It remains to describe the **pyramid type scheme** for the wavelet transform. Since the trial functions and test functionals are defined by a parametrization, the transforms over Γ reduce to the corresponding wavelet transforms over the half axis. Let us first consider the inverse wavelet transform for the **test functionals**. Suppose that, for f given on $[-\infty, 0]$, the values $\{\vartheta_k^l(f),\ k = 1,\dots,N_l^T,\ l = 0,\dots,lev\}$ are known. We have to determine the values $\{f(t_k),\ k = 1,\dots,N\}$. To get these, we successively compute the values $\{f(t_k^l),\ k = 1,\dots,N_l^T\}$, $l = 0,\dots,lev$ (cf. (1.11)). Clearly, the values $\{f(t_k^0) = \vartheta_k^0(f),\ k = 1,\dots,N_0^T\}$ are given. We get the values $\{f(t_k^1),\ k = 1,\dots,N_1^T\}$ by (cf.(1.12))

$$f(t_k^1) = \vartheta_k^1(f) + \sum_{j=1}^{2}\alpha_{k,j}^1 f(t_{k,j}^1),$$

where the values $f(t_{k,j}^1)$ belong to the given sequence $\{f(t_k^0),\ k = 1,\dots,N_0^T\}$. Knowing $\{f(t_k^1),\ k = 1,\dots,N_1^T\}$, we compute $\{f(t_k^2),\ k = 1,\dots,N_2^T\}$ by (cf.(1.12))

$$f(t_k^2) = \vartheta_k^2(f) + \sum_{j=1}^{2}\alpha_{k,j}^2 f(t_{k,j}^2),$$

where $f(t_{k,j}^2)$ is taken from the just computed sequence $\cup_{l=0,1}\{f(t_k^l),\ k = 1,\dots,N_l^T\}$. Similarly, we compute $\{f(t_k^3),\ k = 1,\dots,N_3^T\}$ from $\{\vartheta_k^3(f),\ k = 1,\dots,N_3^T\}$ and from $\cup_{l=0,1,2}\{f(t_k^l),\ k = 1,\dots,N_l^T\}$. Following this procedure we finally compute $\{f(t_k^{lev}),\ k = 1,\dots,N_{lev}^T\}$ and arrive at the set of values $\{f(t_k),\ k = 1,\dots,N\} = \cup_{l=0,\dots,lev}\{f(t_k^l),\ k = 1,\dots,N_l^T\}$.

To describe the **pyramid type scheme for the wavelet transform in the trial space**, we suppose that the function $z_N = \sum_{k=1,\dots,N}\xi_k\varphi_k$ over $[-\infty, 0]$ is given and seek the coefficients η_k^l of the representation $z_N = \sum_{l=0}^{lev}\sum_{k=1}^{N_l^A}\eta_k^l\psi_k^l$. Let us set $N_{lev}^S := N$, denote the spline basis function φ_k by φ_k^{lev}, and introduce (compare Sect.1.1) $\varphi_k^l(s) := \varphi(s/(h2^{lev-l})+k-1)$, $k=1,\dots,N_l^S-1$, $\varphi_{N_l^S}^l(s) := 1-\sum_{k=1}^{N_l^S-1}\varphi_k^l(s)$ with $N_l^S := 7\cdot 2^l+1$ and $l=0,1,\dots,lev-1$. Clearly, the spaces $V_l := \mathit{span}\{\varphi_k^l,\ k=1,\dots,N_l^S\}$ satisfy $V_0 \subseteq V_1 \ldots \subseteq V_{lev}$. Beside the basis $\{\varphi_k^l,\ k=1,\dots,N_l^S\}$ also the system $\{\varphi_k^{l-1},\ k=1,\dots,N_{l-1}^S\}\cup\{\psi_k^l,\ k=1,\dots,N_l^A\}$ forms a basis of V_l, $l=1,\dots,lev$. Moreover, we get so called two-scale relations (cf.[14] and (1.15), (1.17))

$$
\begin{aligned}
\varphi_1^{l-1} &= \varphi_1^l + \frac{1}{2}\varphi_2^l,\\
\varphi_k^{l-1} &= \varphi_{2k-1}^l + \frac{1}{2}\varphi_{2k}^l + \frac{1}{2}\varphi_{2k-2}^l, \quad k=2,\dots,N_l^S-1,\\
\varphi_{N_{l-1}^S}^{l-1} &= \varphi_{N_l^S}^l + \frac{1}{2}\varphi_{N_l^S-1}^l,\\
\psi_1^l &= -\varphi_2^l + \frac{1}{2}\varphi_3^l + \varphi_1^l,\\
\psi_k^l &= -\varphi_{2k}^l + \frac{1}{2}\varphi_{2k+1}^l + \frac{1}{2}\varphi_{2k-1}^l, \quad k=2,\dots,N_l^A-1,\\
\psi_{N_l^A}^l &= -\varphi_{N_l^S-1}^l + \varphi_{N_l^S}^l + \frac{1}{2}\varphi_{N_l^S-2}^l
\end{aligned}
\tag{1.27}
$$

valid for $l=1,\dots,lev$. Now let us denote the basis transform mapping the coefficients $\{\xi_k^{l-1},\ k=1,\dots,N_{l-1}^S\}\cup\{\eta_k^l,\ k=1,\dots,N_l^A\}$ of $f=\sum_{k=1}^{N_{l-1}^S}\xi_k^{l-1}\varphi_k^{l-1}+\sum_{k=1}^{N_l^A}\eta_k^l\psi_k^l$ onto the coefficients $\{\xi_k^l,\ k=1,\dots,N_l^S\}$ of $f=\sum_{k=1}^{N_l^S}\xi_k^l\varphi_k^l$ by T_l. From (1.27) we infer that the matrix of T_l mapping the vector $\{\xi_1^{l-1},\eta_1^l,\xi_2^{l-1},\eta_2^l,\xi_3^{l-1},\dots,\eta_{N_{l-1}^S-1}^l,\xi_{N_{l-1}^S}^{l-1}\}$ onto $\{\xi_1^l,\xi_2^l,\dots,\xi_{N_l^S}^l\}$ is tridiagonal. Hence, the application of T_l requires $O(N_l^S)$ arithmetic operations. Even the inverse transform applied to a vector can be computed with $O(N_l^S)$ operations using tridiagonal solvers. Now the pyramid type scheme looks as follows. Applying the inverse of T_{lev} including the tridiagonal solver to the given vector $\{\xi_k^{lev} := \xi_k,\ k=1,\dots,N_{lev}^S\}$ of $z_N=\sum_{k=1}^N\xi_k\varphi_k$, we compute $\{\eta_k^{lev},\ k=1,\dots,N_{lev}^A\}$ and $\{\xi_k^{lev-1},\ k=1,\dots,N_{lev-1}^S\}$. Next we apply the inverse of T_{lev-1} to $\{\xi_k^{lev-1},\ k=1,\dots,N_{lev-1}^S\}$ and get $\{\eta_k^{lev-1},\ k=1,\dots,N_{lev-1}^A\}$ as well as $\{\xi_k^{lev-2},\ k=1,\dots,N_{lev-2}^S\}$. Similarly we proceed until the application of T_1 to $\{\xi_k^1,\ k=1,\dots,N_1^S\}$ and obtain $\{\eta_k^1,\ k=1,\dots,N_1^A\}$ as well as $\{\xi_k^0,\ k=1,\dots,N_0^S\}$. Since

$$
\begin{aligned}
\sum_{k=1}^{N}\xi_k^{lev}\varphi_k^{lev} &= \sum_{k=1}^{N_{lev}^A}\eta_k^{lev}\psi_k^{lev} + \sum_{k=1}^{N_{lev-1}^S}\xi_k^{lev-1}\varphi_k^{lev-1}\\
&= \sum_{k=1}^{N_{lev}^A}\eta_k^{lev}\psi_k^{lev} + \sum_{k=1}^{N_{lev-1}^A}\eta_k^{lev-1}\psi_k^{lev-1} + \sum_{k=1}^{N_{lev-2}^S}\xi_k^{lev-2}\varphi_k^{lev-2}\\
&= \sum_{l=1}^{lev}\sum_{k=1}^{N_l^A}\eta_k^l\psi_k^l + \sum_{k=1}^{N_0^S}\xi_k^0\varphi_k^0
\end{aligned}
$$

and since $\psi_k^0=\varphi_k^0$, $\eta_k^0=\xi_k^0$, the computed coefficients $\{\eta_k^l,\ k=1,\dots,N_l^A,\ l=0,\dots,lev\}$ represent the wavelet transform.

1.5 Piecewise cubic collocation

The algorithm with piecewise cubic spline functions in the trial space looks quite similar to the piecewise linear collocation. Analogously to the notation from Sects.1.1-1.4, we introduce the collocation points by

$$\begin{aligned} N &:= 7 \cdot 2^{lev} + 1, \quad h := \zeta \log N / N, \\ t_1 &:= 0, \; t_2 := -h/2, \; t_k := -(k-2)h, \; k = 3, \ldots, N-1, \; t_N := -\infty, \\ P_\iota &:= P_{(j_\iota, k_\iota)} := \Phi_{j_\iota}(t_{k_\iota}), \; \iota \in I. \end{aligned} \tag{1.28}$$

By φ we now denote the cubic B-spline such that $supp\, \varphi = [-2, 2]$, that φ is continuously differentiable, that the integral of φ is one, and that the restriction $\varphi|_{[k,k+1]}$, $k = -2, -1, 0, 1$ is a cubic polynomial. We set $\varphi_k(s) := \varphi(s/h + k - 2)$, $k = 1, \ldots, N-1$ and $\varphi_N(s) := 1 - \sum_{k=1}^{N-1} \varphi_k(s)$. Thus the basis functions in our cubic trial space over Γ are given by

$$\varphi_\iota(\Phi_m(s)) := \varphi_{(j_\iota,k_\iota)}(\Phi_m(s)) := \begin{cases} \varphi_{k_\iota}(s) & \text{if } j_\iota = m \\ 0 & \text{else }, \end{cases} \qquad \iota \in I. \tag{1.29}$$

Using this notation, the cubic collocation method is the method (1.4). For the discretization of the cubic spline collocation we use the quadrature (1.5),(1.6), where now $Q_1(f; 0, e^{-(N-1)h})$ and $Q_2(f; -(N-1)h, 0)$ denote the composite Simpson rule over the same partitions (I.e., we take the points of the partitions in Sect.1.2 and the midpoints of each subinterval as quadrature nodes.) as in Sect.1.2. The quadrature rule (1.6) with mesh size $h_{qu} = h \cdot 2^l$ is the rule, where $Q_2(f; -(N-1)h, 0)$ is Simpson's rule applied to $[-(N-1)h, 0] \ni s \mapsto f(e^s)e^s$ over the partition $Part$ of $[-(N-1)h, 0]$ with

$$\begin{aligned} Part &:= \Big\{ -kh_{qu} : \; k = 0, \ldots, 2^{-l}(N-1) \Big\} \bigcup \\ &\bigcup_{m=0,\ldots,l-1} \Big\{ -k(h2^m) : \; k = 0, \ldots, 2[co_0 + co_1 lev] + 3 \Big\} \bigcup \\ &\qquad \bigcup_{m=0,\ldots,l-1} \Big\{ -k(h2^m) : \; k = 2^{-m}(N-1) - 7, \ldots, 2^{-m}(N-1) \Big\}. \end{aligned} \tag{1.30}$$

Here co_0 and co_1 denote suitable non-negative constants. Using the quadrature rules with minimal mesh size h, we get the corresponding discretized collocation by (1.8).

In order to define our wavelet algorithm let us introduce the wavelet test and trial functions. We introduce the partition $\{t_k, \; k = 1, \ldots, N\} = \cup_{l=0,\ldots,lev} \{t_k^l, \; k = 1, \ldots, N_l^T\}$ by

$$\begin{aligned} t_1^0 &:= 0, \; t_2^0 := -h/2, \; t_k^0 := -(k-2)h2^{lev}, \; k = 3, \ldots, N_0^T - 2, \; t^0_{N_0^T - 1} := t_{N-1}, \\ &\qquad t^0_{N_0^T} := -\infty, \\ t_k^l &:= -(2k-1)h2^{lev-l}, \; k = 1, \ldots, N_l^T, \; l = 1, \ldots, lev. \end{aligned} \tag{1.31}$$

The numbers N_l^T are chosen such that $t^0_{N_0^T-2} > t^0_{N_0^T-1} = t_{N-1} \geq -(N_0^T - 3)h2^{lev}$ and $t^l_{N_l^T} > t_{N-1} \geq -(2N_l^T + 1)h2^{lev-l}$, $l = 1, \ldots, lev$ is satisfied. For $l = 0$, we set $\vartheta_k^0 := \delta_{t_k^0}$, $k = 1, \ldots, N_0^T$, and, for $l > 0$,

$$\vartheta_k^l := \delta_{t_k^l} - \sum_{j=1}^{4} \alpha_{k,j}^l \delta_{t_{k,j}^l}, \tag{1.32}$$

where $t_{k,j}^l$, $j = 1, \ldots, 4$ are the four grid points of the coarser levels $\cup_{m=0,\ldots,l-1}\{t_k^m : k = 1, \ldots, N_m^T\}$ nearest to t_k^l. In other words,

$$\begin{aligned} t_{k,1}^l &:= -h2^{lev-(l-1)} \cdot \begin{cases} (k-3) & \text{if } \; -h2^{lev-(l-1)}(k+1) < t_{N-1} \leq -h2^{lev-(l-1)}k \\ (k-4) & \text{if } \; -h2^{lev-(l-1)}k < t_{N-1} \\ (k-1) & \text{if } \; k = 1 \\ (k-2) & \text{else } \; , \end{cases} \\ t_{k,2}^l &:= t_{k,1}^l - h2^{lev-(l-1)}, \;\; t_{k,3}^l := t_{k,1}^l - 2h2^{lev-(l-1)}, \;\; t_{k,4}^l := t_{k,1}^l - 3h2^{lev-(l-1)}, \end{aligned} \tag{1.33}$$

The coefficients $\alpha_{k,j}^l$ are chosen such that ϑ_k^l vanishes at all cubic polynomials, i.e., we define

$$\begin{aligned} \alpha_{k,1}^l &:= \begin{cases} -5/16 & \text{if } \; -h2^{lev-(l-1)}k < t_{N-1} \\ 1/16 & \text{if } \; -h2^{lev-(l-1)}(k+1) < t_{N-1} \leq -h2^{lev-(l-1)}k \\ 5/16 & \text{if } \; k = 1 \\ -1/16 & \text{else } \; , \end{cases} \\ \alpha_{k,2}^l &:= \begin{cases} 21/16 & \text{if } \; -h2^{lev-(l-1)}k < t_{N-1} \\ -5/16 & \text{if } \; -h2^{lev-(l-1)}(k+1) < t_{N-1} \leq -h2^{lev-(l-1)}k \\ 15/16 & \text{if } \; k = 1 \\ 9/16 & \text{else } \; , \end{cases} \\ \alpha_{k,3}^l &:= \begin{cases} 35/16 & \text{if } \; -h2^{lev-(l-1)}k < t_{N-1} \\ 15/16 & \text{if } \; -h2^{lev-(l-1)}(k+1) < t_{N-1} \leq -h2^{lev-(l-1)}k \\ -5/16 & \text{if } \; k = 1 \\ 9/16 & \text{else } \; , \end{cases} \\ \alpha_{k,4}^l &:= \begin{cases} -35/16 & \text{if } \; -h2^{lev-(l-1)}k < t_{N-1} \\ 5/16 & \text{if } \; -h2^{lev-(l-1)}(k+1) < t_{N-1} \leq -h2^{lev-(l-1)}k \\ 1/16 & \text{if } \; k = 1 \\ -1/16 & \text{else } \; . \end{cases} \end{aligned} \tag{1.34}$$

Let us turn to the trial functions. Analogously to (1.15) and (1.17) we introduce

$$\psi(s) := \frac{1}{8}\sum_{j=0}^{4}(-1)^j \begin{pmatrix} 4 \\ j \end{pmatrix} \varphi(s-j) \tag{1.35}$$

and set

$$\psi_1^0 \;\; := \;\; \varphi(s/(h2^{lev})), \tag{1.36}$$

$$
\begin{aligned}
\psi_2^0 &:= \varphi(s/(h2^{lev})+1)+\varphi(s/(h2^{lev})-1),\\
\psi_k^0(s) &:= \varphi(s/(h2^{lev})+k-1),\ k=3,\dots,N_0^A-1,\ N_0^A:=7,\\
\psi_{N_0^A}^0(s) &:= \begin{cases} \sum_{k=N_0^A}^{N_0^A+2}\varphi(s/(h2^{lev})+k-1) & \text{if } s\geq -(N-1)h\\ 1 & \text{if } s< -(N-1)h\end{cases}\\
\psi_1^l(s) &:= \psi(s/(h2^{lev-l})+3)+\psi(s/(h2^{lev-l})+1),\\
\psi_2^l(s) &:= \psi(s/(h2^{lev-l})+5)+\psi(s/(h2^{lev-l})-1),\\
\psi_k^l(s) &:= \psi(s/(h2^{lev-l})+(2k+1)),\ \ k=3,\dots,N_l^A-2,\ \ N_l^A:=7\cdot 2^{l-1},\\
\psi_{N_l^A-1}^l(s) &:= \begin{cases} \psi(s/(h2^{lev-l})+(2N_l^A-1))+ & \text{if } s\geq -(N-1)h\\ \quad\psi(s/(h2^{lev-l})+(2N_l^A+5))+ & \\ \frac{1}{8}\varphi(s/(h2^{lev-l})-2N_l^A) & \\ 1/8 & \text{if } s< -(N-1)h\end{cases}\\
\psi_{N_l^A}^l(s) &:= \begin{cases} \psi(s/(h2^{lev-l})+(2N_l^A+1))+ & \text{if } s\geq -(N-1)h\\ \quad\psi(s/(h2^{lev-l})+(2N_l^A+3))+ & \\ \frac{7}{8}\varphi(s/(h2^{lev-l})-2N_l^A) & \\ 7/8 & \text{if } s< -(N-1)h\end{cases}\\
& \qquad\qquad l=1,\dots,lev-1\\
\psi_1^{lev}(s) &:= \varphi(s/h-1),\\
\psi_2^{lev}(s) &:= \psi(s/h+3)+\psi(s/h+1),\\
\psi_3^{lev}(s) &:= \psi(s/h+5)+\psi(s/h-1),\\
\psi_k^{lev}(s) &:= \psi(s/h+(2k-1)),\ \ k=4,\dots,N_{lev}^A-2,\ \ N_{lev}^A:=7\cdot 2^{lev-1}+1,\\
\psi_{N_{lev}^A-1}^{lev}(s) &:= \begin{cases} \psi(s/h+(2N_{lev}^A-3))+ & \text{if } s\geq -(N-1)h\\ \quad\psi(s/h+(2N_{lev}^A+3))+ & \\ \frac{1}{8}\varphi(s/h-2N_{lev}^A-2) & \\ 1/8 & \text{if } s< -(N-1)h\end{cases}\\
\psi_{N_{lev}^A}^{lev}(s) &:= \begin{cases} \psi(s/h+(2N_{lev}^A-1))+ & \text{if } s\geq -(N-1)h\\ \quad\psi(s/h+(2N_{lev}^A+1))+ & \\ \frac{7}{8}\varphi(s/h-2N_{lev}^A-2) & \\ 7/8 & \text{if } s< -(N-1)h\ .\end{cases}
\end{aligned}
$$

Now we define ψ_ι and $\hat{P}_\kappa$ by (1.18) and (1.19), respectively. Analogously to the beginning of Sect.1.4 we get $f(\hat{P}_\kappa)=\sum_{i=1}^5 \alpha_i f(P_{\kappa,i})$ with appropriate α_i and $P_{\kappa,i}$. For a fixed $\kappa\in I$, the set $I^A(\hat{P}_\kappa)$ of indices for which the entry $(A\psi_\iota)(\hat{P}_\kappa)$ of B_N is not neglected in the compression step is now introduced as follows. An index $\iota=(j_\iota,k_\iota,l_\iota)\in I^A$ belongs to $I^A(\hat{P}_\kappa)$ if $l_\iota\leq lev-l_\kappa$ or if $\psi_\iota(P_{\kappa,i})\neq 0$ with $i=1,\dots,5$ or if ψ_ι is a boundary wavelet or if $k_\iota\leq co_0+co_1 lev$. Using this new set $I^A(\hat{P}_\kappa)$, the wavelet algorithm with cubic trial functions is the same as that presented in Sect.1.4. It leads to a compressed stiffness matrix with a number of non-zero entries less than a constant times N times a power of $\log N$. The number of necessary arithmetic operations in the algorithm is also less than a constant times N times a power of $\log N$.

Let us remark that, for our choice of wavelets in the trial space, the compression $x_N^c := \sum_{\iota\in I^A(\hat{P}_\kappa)}\xi_\iota\psi_\iota$ of a smooth cubic spline $x_N=\sum_{\iota\in I^A}\xi_\iota\psi_\iota$ is not smooth in the neighbourhood of the points $Q^j=P_{(j,0)}$ if $co_1=co_0=0$. In fact, the introduction of ψ_1^{lev} instead of a basis function $\varphi(s/(h2^{lev})-1)$ on level zero ensures the boundedness of the wavelet transform but leads to non-smoothness in the neighbourhoods of the midpoints

$Q^j = P_{(j,0)}$ of the sides of Γ. In order to compensate this effect we have introduced the constants co_0, co_1.

2 NUMERICAL TESTS

For a numerical example, we take the equilateral triangle $\Omega = \triangle ABC$ with corner points $A := (-1/2, 0)$, $B := (1/2, 0)$, and $C := (0, \sqrt{3}/2)$. We consider the harmonic function $U(P) := U(s_P, t_P) := \log \sqrt{(s_P - 0.1)^2 + (t_P - e - 0.2)^2}$ and get

$$U(P) = \frac{1}{2} \int_\Gamma k(P, Q) x(Q) d_Q \Gamma, \quad P \in \Omega, \tag{2.1}$$

where x is the solution of $Ax = y := 2U|_\Gamma$. In accordance with Sect.1.1 we divide the boundary Γ into $K = 6$ equal parts and determine an approximate solution x_N of x by the algorithm of Sect.1.5. We compute, for $P_1 = (0.1, 0.2)$, the approximation

$$U_N(P_1) = \frac{1}{2} \sum_{\mu \in J} k(P_1, Q_\mu) x_N(Q_\mu) \omega_\mu \tag{2.2}$$

of $U(P_1) = 1$. By DE_N we denote the **error of the Dirichlet solution** $|U_N(P_1) - U(P_1)|$ and by $SE_{N'}$ the **supremum norm error of the solution for the integral equation** $\|x_N - x_{N'}\|_{L^\infty} \sim \|x - x_{N'}\|_{L^\infty}$ (An approximate value of this supremum is computed by a maximum over a large number of points of Γ.), where $N := 7 \cdot 2^{lev} + 1$ and $N' := 7 \cdot 2^{lev-1} + 1$. Furthermore, we determine the approximate value $\gamma_N := [\log SE_N - \log SE_{N'}]/[\log h_N - \log h_{N'}]$ with $h_N := \zeta \log N/N$ and $h_{N'} := \zeta \log N'/N'$ for the order γ of the error $SE_N \sim h_N^\gamma$. In Table 1 (cf. also Figure 4) we present the corresponding numerical results. These results show that, for an approximate solution U_N of the Dirichlet problem away from the boundary $\Gamma := \partial\Omega$, a small mesh parameter ζ is sufficient. We observe a convergence rate $DE_N \sim h_N^4$ if $\zeta = 1$. The error DE_N is larger for $\zeta > 1$. However, we conjecture that the results for larger ζ can be improved if a better quadrature rule is applied in (2.2). Since we are interested in an approximation of U over the whole of Ω and since this error can be estimated by the supremum norm, we are mainly interested in SE_N and not in DE_N. We compute DE_N only to demonstrate the closeness of x_N to x. For the supremum error, we remark that the function x has an asymptotic behaviour of $x(s, 0) - x(-1/2, 0) \sim (s + 1/2)^{3/5}$ if $s \longrightarrow -1/2$ (cf. Sect.4 and [37]). Hence, we expect $\gamma_N \sim \min(4, \zeta 3/5)$ (cf. Corollary 4.2 and Remark 4.3). Table 1 seems to confirm this asymptotic rate.

Now let us consider the compression properties. The **compression rate** CR is the quotient of the number of non-zero entries of B'_N per number of all entries ${N_u}^2$, where $N_u := 6 \cdot N$ is the number of equations in the collocation system (1.4) and N is the number of collocation points over each part Γ_j, $\ j = 1, \ldots, 6$. The compression algorithm of Sects.1.4 and 1.5 has been established to obtain a compression error of order $O(h^4 [\log h^{-1}]^\mu)$, where μ denotes a certain non-negative constant. Since the approximation error without compression is of order $O(h^{\min(4, \zeta 3/5)} [\log h^{-1}]^\mu)$, a better compression is possible. Thus we introduce a parameter ρ with $1 \geq \rho > 0$ and define $I^A(\hat{P}_\kappa)$ to be the set of all $\iota \in I^A$ such that $l_\iota \leq \rho \cdot lev - l_\kappa$ or that $\psi_\iota(P_{\kappa,i}) \neq 0$ with $i = 1, \ldots, 5$ or that ψ_ι is a boundary wavelet or that $k_\iota \leq co_0 + co_1 lev$. Analogously to the estimates of Sect.4, we get a compression error of $O(h^{4\rho} [\log h^{-1}]^\mu)$. Consequently, we can choose

ζ	lev	$N_u = 6 \cdot N$	SE_N	γ_N	DE_N
1	0	49	0.089		0.000027
	1	90	0.058	0.99	0.0000050
	2	174	0.036	0.96	0.00000098
	3	342	0.023	0.83	0.00000015
	4	678	0.015	0.80	0.000000017
	5	1350	0.0094	0.78	0.0000000015
	6	2694	0.0065	0.62	0.00000000016
	7	5283	0.0042	0.76	0.0000000000069
	8	10758	0.0027	0.75	0.00000000000063
2	0	49	0.035		0.000075
	1	90	0.014	2.02	0.000010
	2	174	0.0058	1.85	0.0000048
	3	342	0.0024	1.66	0.00000097
	4	678	0.00099	1.59	0.0000014
	5	1350	0.00041	1.55	0.00000060
	6	2694	0.00018	1.40	0.00000046
	7	5283	0.000080	1.36	0.00000060
	8	10758	0.000033	1.48	0.000000021
	9	21510	0.000015	1.22	0.000000028
	10	43014			0.000000034
3	0	49	0.013		0.00023
	1	90	0.0035	3.08	0.000074
	2	174	0.00088	2.77	0.0000060
	3	342	0.00023	2.47	0.0000013
	4	678	0.000063	2.35	0.00000063
	5	1350	0.000017	2.32	0.00000012
	6	2694	0.0000048	2.15	0.000000071
	7	5283	0.0000014	2.08	0.000000011
	8	10758	0.00000058	1.48	0.0000000027
	9	21510	0.00000018	1.87	0.00000000071
	10	43014			0.000000000049
4	0	49	0.005		0.000055
	1	90	0.00082	4.09	0.000010
	2	174	0.00013	3.71	0.0000010
	3	342	0.000022	3.20	0.00000056
	4	678	0.0000040	3.26	0.0000026
	5	1350	0.00000077	2.88	0.000000035
	6	2694	0.00000052	0.69	0.00000020

Table 1: Approximation properties of the algorithm.

$\rho = 0.375$ for $\zeta = 2$ and $\rho = 0.6$ for $\zeta = 3$. Moreover, in our numerical examples we choose $lev_0 = 0$. This leads to smaller powers of $\log N$ in the estimates. Though the stability proof fails for $lev_0 = 0$, we have not observed any instability. In the Tables 2 and 3 (cf. also Figures 5 and 6) we present the compression rates, the **computation time TW in CPU seconds for the assemblation of the compressed matrix B'_N,**

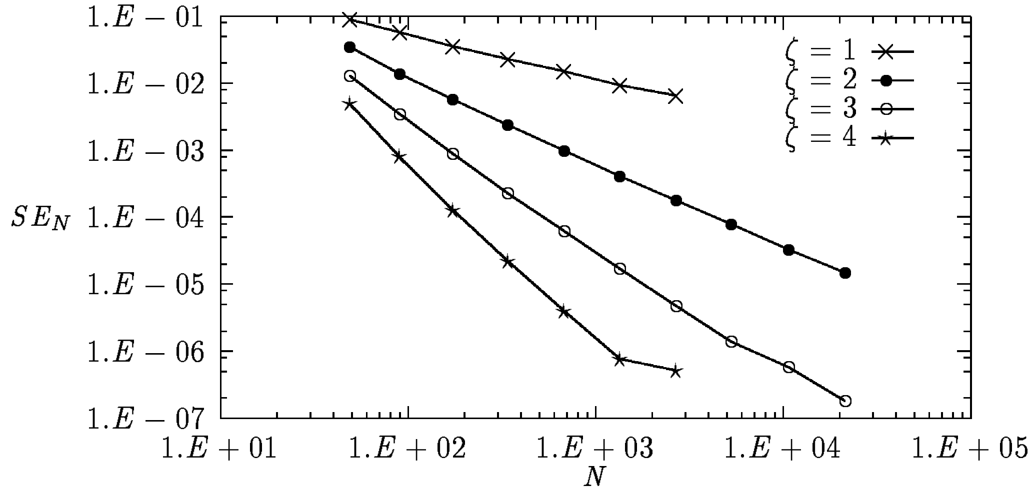

Figure 4: Orders of convergence

lev	$N_u = 6 \cdot N$	CR	TW	T
0	49	1.00	0.26	0.16
1	90	0.93	1.05	0.63
2	174	0.62	3.18	2.20
3	342	0.43	9.71	8.88
4	678	0.25	23.87	34.06
5	1350	0.16	56.06	136.15
6	2694	0.086	141.06	548.58
7	5382	0.048	320.47	2191.87
8	10758	0.027	775.11	
9	21510	0.015	1721.56	
10	43014	0.0079	3775.10	

Table 2: Compression rates and computing time for $\zeta = 2$, $\rho = 0.375$, and $co_0 = 0 = co_1$

and the time T for the computation of the corresponding matrix A'_N (cf. Sect.1.2). Note that the most time consuming part of the computation is that for the computation of the stiffness matrix. It turns out that the computation time T grows by factor four if the dimension $N_u = 6 \cdot N$ of the linear system is doubled. The time TW grows by a factor between 2.5 and 3. For $\zeta = 2$, the wavelet algorithm is faster if the number of levels lev is greater or equal to four. Since our computer has a main memory of 512 MB, we had to restrict our computations without wavelets to at most seven levels. The compression algorithm allows us to go up to ten levels. For $\zeta = 1$ and the small errors DE_N presented in Table 1, the compression parameters of the wavelet algorithm should be chosen as in Table 3 and the resulting computing time is similar. Finally, let us mention that we have tested also a boundary curve, where one straight line segment of the triangle is replaced

lev	$N_u = 6 \cdot N$	*CR*	*TW*	*T*
0	49	1.00	0.26	0.18
1	90	0.93	1.11	0.57
2	174	0.79	4.25	2.15
3	342	0.48	12.43	8.29
4	678	0.34	34.86	32.73
5	1350	0.21	93.57	130.59
6	2694	0.13	265.35	524.06
7	5382	0.073	655.73	2111.37
8	10758	0.044	1585.44	
9	21510	0.025	3809.31	
10	43014	0.015	10544.19	

Table 3: Compression rates and computing time for $\zeta = 3$, $\rho = 0.6$, and $co_0 = 2$, $co_1 = 0.5$

by a sine shaped arc. The obtained results have turned out to be quite similar.

All the computations have been performed on a DEC 3000 AXP 500 workstation.

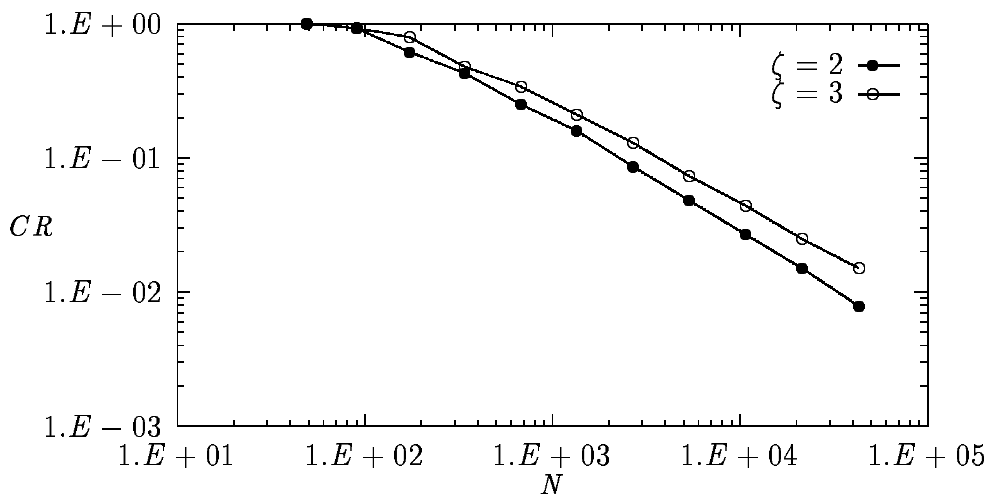

Figure 5: Compression rates

3 STABILITY OF THE METHOD

Let us consider the operator equation $Ax = y$ in the space $C(\Gamma)$ of all bounded and piecewise continuous functions over Γ which are continuous on each straight line segment Γ_j. Clearly, there is a constant C (Here and in the following we denote by C a non-negative constant which varies from instance to instance.) such that, for any sequence $\{\xi_\iota\}_{\iota\in I}$ of real numbers,

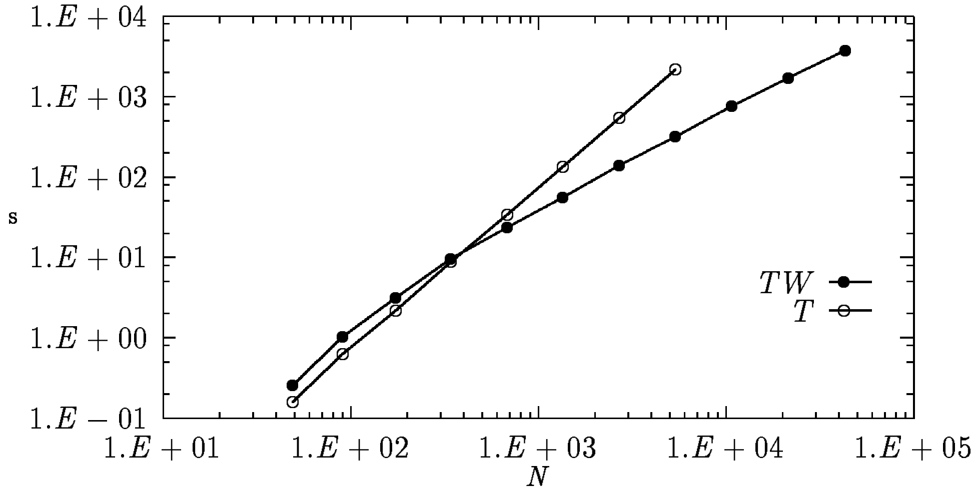

Figure 6: Time for the computation of the matrix in CPU-seconds, $\zeta = 2$

$$\frac{1}{C}\|\sum_{\iota\in I}\xi_\iota\varphi_\iota\|_{L^\infty} \le \sup_{\iota\in I}|\xi_\iota| \le C\|\sum_{\iota\in I}\xi_\iota\varphi_\iota\|_{L^\infty}. \tag{3.1}$$

Hence, we have to consider the approximate operators A_N and A'_N in the space $\mathcal{L}(l^\infty(I))$ of bounded linear operators over the space $l^\infty(I)$ of bounded sequences $\{\xi_\iota\}_{\iota\in I}$. From the boundedness of $A \in \mathcal{L}(C(\Gamma))$ it is not hard to see that A_N is uniformly bounded with respect to N. Now the sequence $\{A_N\}$ and the corresponding collocation method (1.4) is called stable if $A_N \in \mathcal{L}(l^\infty(I))$ is invertible for N large enough and if $(A_N)^{-1}$ is uniformly bounded with respect to N. It is well known that the derivation of the stability is the main part in the proof of optimal convergence rates for the collocation.

THEOREM 3.1 *The piecewise linear collocation method (1.4) is stable. Moreover, the discretized collocation (cf. (1.8)) is stable if only the quadrature parameter i_* (cf. the definition of $Q_1(f;0,e^{-(N-1)h})$ in Sect.1.2) is large enough.*

PROOF: There exist several methods for proving Theorem 3.1 (cf. e.g. [11, 38, 13, 3, 44, 26, 36, 45, 42]). Therefore, we shall give only some ideas and references without going into details. In any case, we sketch a proof which can be applied also for piecewise cubic trial functions.

It is a well-known fact that localization techniques apply to the stability theory of numerical methods for operators of local type (cf. [35, 54, 41, 42]). This allows us to restrict our consideration to the simpler case of a curve Γ equal to the boundary of a plane sector consisting of two half axis. It is not hard to show that in this special case the matrix A_N takes the form

$$\begin{pmatrix} I & K_N \\ K_N & I \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} I+K_N & 0 \\ 0 & I-K_N \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}, \tag{3.2}$$

where K_N stands for the discretized double layer operator acting from one half axis of Γ onto the other. Consequently, it remains to prove the stability of $I \pm K_N$. Following [47],

it is not hard to see that K_N is a Toeplitz operator the symbol of which is differentiable and satisfies $\|symbol_N\| \leq q < 1$. Hence, $\{A_N\}$ is stable.
Moreover, following part a) of the proof to Theorem 4.2 in [47] one easily gets stability for the discretized method. ∎

REMARK 3.2 *Theorem 3.1 holds also for the piecewise cubic collocation.*

Our next concern is to prove stability also for the approximate operator corresponding to the wavelet algorithm of Sect.1.4. This operator is the one used in the multiplication step of the iteration process, i.e., $A_N^w = Tr_N^T B_N' Tr_N^A$, where B_N' is the transformed, compressed, and discretized stiffness matrix (cf. Sect.1.4), Tr_N^T is the wavelet transform in the space of test functionals, and Tr_N^A stands for the wavelet transform in the trial space. In other words, Tr_N^A maps the vector $\{\xi_\iota\}_{\iota\in I}$ of coefficients of the function $z_N = \sum_{\iota\in I}\xi_\iota\varphi_\iota$ into the vector of coefficients of the same function z_n with respect to the wavelet basis $\{\psi_\iota\}_{\iota\in I^A}$. For any continuous function f over Γ, the transform Tr_N^T maps the vector $\{f(\hat{P}_\kappa)\}_{\kappa\in I^T}$ into $\{f(P_\kappa)\}_{\kappa\in I}$. Clearly, in view of (3.1) we have to consider A_N^w in the space $\mathcal{L}(l^\infty(I))$. Let us start our investigations showing the boundedness of the wavelet transforms.

Obviously, the transform Tr_N^T is bounded if the transform T_N^T over the interval $[-\infty, 0]$ mapping $\{\vartheta_k^l(f)\}_{k=1,\dots,N_l^T,\, l=0,\dots,lev}$ to $\{f(t_k)\}_{k=1,\dots,N}$ has this property. Before we consider T_N^T let us introduce the corresponding mapping over the whole axis $\mathbb{R}$. We set $\tilde{t}_k^0 := -(k-1)h2^{lev}$, $k \in \mathbb{Z}$ and $\tilde{t}_k^l := -(2k-1)h2^{lev-l}$, $k \in \mathbb{Z}$, $l = 1,\dots,lev$. Further we introduce the wavelet functionals $\tilde{\vartheta}_k^0(f) := f(t_k^0)$ and $\tilde{\vartheta}_k^l(f) := f(\tilde{t}_k^l) - \frac{1}{2}[f(\tilde{t}_{k,1}^l) + f(\tilde{t}_{k,2}^l)]$ with $l = 1,\dots,lev$, $k \in \mathbb{Z}$ and $\tilde{t}_{k,1}^l := -h2^{lev-(l-1)}(k-1)$, $\tilde{t}_{k,2}^l := -h2^{lev-(l-1)}k$. By T^T we denote the transform $T^T : \{\tilde{\vartheta}_k^l(f)\}_{l=0,\dots,lev,\, k\in\mathbb{Z}} \mapsto \{f(-(k-1)h)\}_{k\in\mathbb{Z}}$. This mapping has just the pyramid form, i.e., setting $\eta^l := \{\eta_k^l\}_{k\in\mathbb{Z}}$, $\xi^l := \{\xi_k^l\}_{k\in\mathbb{Z}}$ with $\eta_k^l := \tilde{\vartheta}_k^l(f)$ and $\xi_k^l := f(-(k-1)h2^{lev-l})$, we get $T^T : \{\xi^0, \eta^1, \dots, \eta^l\} \mapsto \xi^{lev}$ and the two-scale relation (refinement equation)

$$\xi_k^l = \begin{cases} \xi_{s+1}^{l-1} & \text{if } k = 2s+1 \\ \eta_{s+1}^l + [\xi_{s+1}^{l-1} + \xi_{s+2}^{l-1}] & \text{if } k = 2s+2. \end{cases} \tag{3.3}$$

Thus ξ^{lev} can be calculated following the scheme

$$\begin{array}{ccccccccc} \xi^0 & \longrightarrow & \xi^1 & \longrightarrow & \xi^2 & \longrightarrow & \dots & \longrightarrow & \xi^{lev}, \\ & \nearrow & & \nearrow & & & & \nearrow & \\ \eta^1 & & \eta^2 & & & & \eta^{lev} & & \end{array} \tag{3.4}$$

where ξ^0, η^1, ... , η^{lev} are given and ξ^{lev} is to be determined.

LEMMA 3.3 *The wavelet transform $T^T : [l^\infty]^{lev} \longrightarrow l^\infty$ is bounded by $2(lev+1)$.*

PROOF: Identifying ξ^l and η^l with the functions $\xi^l(t) := \sum_{k\in\mathbb{Z}}\xi_k^l t^{k-1}$, $\eta^l(t) := \sum_{k\in\mathbb{Z}}\eta_k^l t^{k-1}$ defined over the unit circle, Equ. (3.3) implies

$$\xi^l(t) = t\eta^l(t^2) + g(t)\xi^{l-1}(t^2), \;\; g(t) := 1 + \frac{1}{2}[t + t^{-1}]. \tag{3.5}$$

Hence,

$$\xi^{lev}(t) \;\; = \;\; \sum_{l=0}^{lev-1} g_l(t)t^{2^l}\eta^{lev-l}(t^{2^{l+1}}) + g_{lev}(t)\xi^0(t^{2^{lev}}), \tag{3.6}$$

$$g_0(t) \;\; := \;\; 1, \quad g_l(t) := g_{l-1}(t^2)g(t). \tag{3.7}$$

Since $g_l(t)t^{2^l} = \sum_{j=1}^{2^{l+1}+1} \lambda_j^l t^j$, we observe that, for a fixed coefficient ξ_k^{lev}, to ξ_k^{lev} there contribute at most two coefficients of each η^{lev-l}, $l = 1, \dots, lev$ and two of ξ^0. Thus

$$|\xi_k^{lev}| \leq 2 \sup_j |\lambda_j^{lev}| \sup_{k \in \mathbb{Z}} |\xi_k^0| + 2 \sum_{l=1}^{lev} \sup_j |\lambda_j^{lev-l}| \sup_{k \in \mathbb{Z}} |\eta_k^l|. \tag{3.8}$$

It remains to check the boundedness of $\sup_j |\lambda_j^l|$. Since the λ_j^l, $j \in \mathbb{Z}$ are the Fourier coefficients of the function g_l, we get $|\lambda_j^l| \leq \int_0^1 |g_l(e^{i2\pi s})| ds$. Consequently, we only have to show that $\int_0^1 |g_l(e^{i2\pi s})| ds = \int_0^1 g_l(e^{i2\pi s}) ds = 1$. We prove this by induction. Clearly, $\int_0^1 g_0(e^{i2\pi s}) ds = 1$. Let us suppose $\int_0^1 g_{l-1}(e^{i2\pi s}) ds = 1$. The symmetry $g_{l-1}(e^{2\pi s}) = g_{l-1}(e^{2\pi(1-s)})$ implies that $g_{l-1}(e^{2\pi s}) = \sum_{j=0}^{\infty} c_j \cos(2\pi j s)$ and $g_{l-1}(e^{2\pi 2s}) = \sum_{j=0}^{\infty} c_j \cos(2\pi 2 j s)$. Thus $g_{l-1}(e^{2\pi 2s})$ is orthogonal to $\cos(2\pi s)$ and we conclude

$$\begin{aligned} \int_0^1 g_l(e^{i2\pi s}) ds &= \int_0^1 g_{l-1}(e^{i2\pi 2s}) g(e^{2\pi s}) ds \\ &= \int_0^1 g_{l-1}(e^{i2\pi 2s}) ds + \int_0^1 g_{l-1}(e^{i2\pi 2s}) \cos(2\pi s) ds \\ &= \int_0^2 g_{l-1}(e^{i2\pi t}) dt/2 = \int_0^1 g_{l-1}(e^{i2\pi t}) dt = 1. \end{aligned} \tag{3.9}$$

In other words, $\sup_{j,l} |\lambda_j^l| \leq 1$ and the proof is finished. ∎

Now let $l^\infty(n)$ stand for the space $\mathbb{R}^n$ supplied with the supremum norm and consider $T_N^T : l^\infty(N_0^T) \oplus l^\infty(N_1^T) \oplus \dots \oplus l^\infty(N_{lev}^T) \longrightarrow l^\infty(N)$.

LEMMA 3.4 *The wavelet transform T_N^T is bounded by $C \cdot lev$, where the constant C is independent of lev and N.*

PROOF: Together with T^T the restriction T_R^T of T^T to $l^\infty(N_0^T) \oplus l^\infty(N_1^T) \oplus \dots \oplus l^\infty(N_{lev}^T) \longrightarrow l^\infty(N)$ is bounded by $C \cdot lev^2$. The difference between T_R^T and T_N^T is that the restricted version

$$\xi_{2N_l^T}^l = \eta_{N_l^T}^l + \frac{1}{2} \xi_{2N_{l-1}^T}^{l-1} \tag{3.10}$$

of relation (3.3) is replaced by (cf. (1.12))

$$\xi_{2N_l^T}^l = \eta_{N_l^T}^l + \frac{3}{2} \xi_{2N_{l-1}^T}^{l-1} - \frac{1}{2} \xi_{2N_{l-1}^T - 1}^{l-1} = \begin{cases} \eta_{N_1^T}^1 + \frac{3}{2} \xi_{2N_0^T}^0 - \frac{1}{2} \xi_{2N_0^T - 1}^0 & \text{if } l = 1 \\ \eta_{N_l^T}^l + \frac{3}{2} \xi_{2N_{l-1}^T}^{l-1} - \frac{1}{2} \xi_{2N_{l-2}^T}^{l-2} & \text{if } l \geq 2. \end{cases} \tag{3.11}$$

If the entries $(T_N^T)_{k,(l,j)}$ of T_N^T are defined by $\xi_k^{lev} = \sum_j (T_N^T)_{k,(0,j)} \xi_j^0 + \sum_{1 \leq l \leq lev} \sum_j (T_N^T)_{k,(l,j)} \eta_j^l$ and the entries of T_R^T similarly, then $(T_N^T)_{k,(l,j)}$ is equal to the entry $(T_R^T)_{k,(l,j)}$ if $j < N_l^T$, $l > 0$ or if $j < N_0^T - 1$, $l = 0$. Indeed, the new relation (3.11) affects only the entries $(T_N^T)_{k,(l,N_l^T)}$ and $(T_N^T)_{k,(0,N_0^T - 1)}$. There are two ways to pass from $\eta_{N_l^T}^l$ to ξ_k^{lev} via (3.3) and (3.11), respectively. If $t_{N_r^T}^r \leq t_k \leq t_{N_{r-1}^T}^r$, then one can go from $\eta_{N_l^T}^l$ to $\xi_{2N_r^T}^r$ using relation (3.11) and from that to ξ_k^{lev} by (3.3) or one goes from $\eta_{N_l^T}^l$ to $\xi_{2N_{r-1}^T}^{r-1}$ using (3.11) and from that to ξ_k^{lev} by (3.3). Let a and b denote the factor by which $\eta_{N_l^T}^l$ is multiplied during the application of (3.11) on the way from $\eta_{N_l^T}^l$ to $\xi_{2N_r^T}^r$ and $\xi_{2N_{r-1}^T}^{r-1}$, respectively. Then we get $(T_N^T)_{k,(l,N_l^T)} = a (T_R^T)_{k,(r,N_r^T)} + b (T_R^T)_{k,(r-1,N_{r-1}^T)}$. Next we shall prove that a and b are bounded. If this is done, then the previous proof implies $|(T_N^T)_{k,(l,j)}| \leq C \sup_{r,s} |(T_R^T)_{k,(r,s)}| \leq C$. Arguing analogously to the previous proof, we also observe that, for each k and l, there are at most two values j with $(T_N^T)_{k,(l,j)} \neq 0$. Thus

$$\|T_N^T\| = \sup_k \sum_{l=0}^{lev} \sum_{j=1}^{N_l^T} |(T_N^T)_{k,(l,j)}| \leq C\, lev\,. \tag{3.12}$$

Let us estimate a and b. If we proceed from $\xi_{N_l^T}^l$ to $\xi_{2N_r^T}^r$ using (3.11), we can choose between a step over two levels with factor $-1/2$ and a step over one level with factor $3/2$. Summing up all products of these possible factors during the way from level l to r, we get a. We observe that $a = a_j$ depends only on the difference $j = r - l$ and that

$$a_j = \frac{3}{2}a_{j-1} - \frac{1}{2}a_{j-2},\ j = 2, 3, \dots\ , \qquad a_0 = 1,\ a_1 = \frac{3}{2}. \tag{3.13}$$

Hence, the values $a = a_j = 2 - 2^{-j}$ are bounded by 2, and $b = a_{j-1}$ is bounded, too. ∎

Now let us consider the wavelet transform Tr_N^A.

LEMMA 3.5 *The wavelet transform Tr_N^A is bounded by a constant independent of lev and N.*

PROOF: Recall the definition of the wavelets in Sect.1.3 (cf.(1.17)). Analogously, to the wavelets in the test space, it suffices to consider the wavelet transform mapping $\{\xi_k\}_{k=1}^N$ to $\{\eta_k^l\}_{l=0,\dots,lev,\ k=1,\dots,N_l^A}$, where $\sum_{k=1}^N \xi_k\varphi_k = \sum_{l=0}^{lev}\sum_{k=1}^{N_l^A} \eta_k^l\psi_k^l$. These wavelets are an adaption of the wavelets over the real axis to the interval $[-\infty, 0]$. However, to any function $z_N = \sum_{k=1}^N \xi_k\varphi_k$ over $[-\infty, 0]$ there corresponds a unique extension $\tilde{z}_N$ over the real axis obtained by the reflections

$$\tilde{z}_N := \begin{cases} \dots & \\ z_N(s - 2(N-1)h) & \text{if } \ (N-1)h \leq s \leq 2(N-1)h \\ z_N(-s) & \text{if } \ 0 \leq s \leq (N-1)h \\ z_N(s) & \text{if } \ -(N-1)h \leq s \leq 0 \\ z_N(-2(N-1)h - s) & \text{if } \ -2(N-1)h \leq s \leq -(N-1)h \\ \dots & . \end{cases} \tag{3.14}$$

The coefficients of z_N with respect to the bases $\{\varphi_k\}$ and $\{\psi_k^l\}$ coincide with those of $\tilde{z}_N$ with respect to the corresponding bases over the real axis (cf. (1.16)). Therefore, it is enough to prove the boundedness of the wavelet transform over the real axis. The corresponding wavelets over the axis are biorthogonal wavelets in the sense of [16]. The dual wavelets, however, have exponential decay instead of finite support. More precisely, setting $h(z) = \sqrt{2}z^{-1}(z+1)^2/4$ and $\tilde{h}(z) = \sqrt{2}[4z^{-1}(z+1)^2]/[z^{-2}(z+1)^4 + z^{-2}(z-1)^4]$ and following the definitions of [16], we get biorthogonal wavelet bases. Indeed, it is not hard to prove that the assumptions of [16], Prop.4.9 are satisfied with $L = 2$ and $k = 1$. Moreover, one can show that the dual scaling function $\tilde{\Phi}$ decays exponentially and is continuous. The wavelet function in this setting is

$$\psi^C(x) = \sqrt{2}\sum_{n\in\mathbb{Z}} (-1)^n \tilde{h}_{-n+1}\Phi(2x + n), \tag{3.15}$$

where $\tilde{h}(z) = \sum_{n\in\mathbb{Z}} \tilde{h}_n z^n$ and Φ is our hat function φ from (1.2). However, $\tilde{h}(z) = g(z)\sqrt{2}z^{-1}(z+1)^2/4$ with $g(z) = 8/[z^{-2} + 6 + z^2]$ such that $g(z) \neq 0$ for $|z| = 1$ and

$g(-z) = g(z) = \sum_{n\in\mathbb{Z}} g_{2n} z^{2n}$. Therefore, the span of translates of the wavelet ψ^C is equal to the span of translates of the wavelet ψ from (1.15). We get the same multiresolution analysis for ψ^C and for ψ. The wavelet coefficients of the wavelet basis defined with ψ^C can be obtained from those defined with ψ by a simple discrete convolution on each level and vice versa. Since ψ is a linear combination of the translates of ψ^C, we conclude that there also exists a dual wavelet ψ^d for our ψ. This ψ^d is continuous, decays exponentially and defines a dual basis $\psi^d_{l,k}(s) := \psi^d(s/(h2^{lev-l}) - (2k-1))/(h2^{lev-l})$, $k \in \mathbb{Z}$, $l = 1,\dots,lev$ with $(\psi^d_{l_1,k_1}, \tilde\psi^l_k) = \delta_{l,l_1}\delta_{k,k_1}$. For $z_N = \sum \eta^l_k \psi^l_k$ and its extension $\tilde z_N = \sum \eta^l_k \tilde\psi^l_k$ (cf. (3.14)), we conclude

$$
\begin{aligned}
|\eta^l_k| &= |(\tilde z_N, \psi^d_{l,k})| \le \|\psi^d_{l,k}\|_{L^1} \|\tilde z_N\|_{L^\infty}, \\
sup_{l,k}|\eta^l_k| &\le C\, \|z_N\|_{L^\infty}.
\end{aligned} \tag{3.16}
$$

Using this, $z_N = \sum \eta^l_k \psi^l_k = \sum \xi_k \varphi_k$, and (3.1), we arrive at the boundedness of the wavelet transform Tr^A_N. ∎

THEOREM 3.6 *The approximate operator $A^w_N := Tr^T_N B'_N Tr^A_N$ of the wavelet algorithm from Sect.1.4 is stable. More precisely, A^w_N is a small perturbation of the stable (cf. Theorem 3.1) approximate operator $A_N = Tr^T_N B_N Tr^A_N$ (cf. the end of Sect.1.3) of the collocation method and there holds:*

i) $\|A_N - Tr^T_N B^c_N Tr^A_N\| \le C \cdot h^2 \cdot lev^4$,

ii) $\|A^w_N - Tr^T_N B^c_N Tr^A_N\| \le C/lev$.

PROOF: Clearly, the stability of A_N and $\|A_N - A^w_N\| \longrightarrow 0$ for $N \longrightarrow \infty$ imply the stability of A^w_N. Thus the stability of A^w_N follows from Theorem 3.1 and the assertions i) and ii). Let us prove the estimate for the compression error in i).
We have to estimate the kernel function $k(P,Q)$ for $P \in supp\, \hat P_\kappa$ and $Q \in supp\, \psi_\iota$, where ψ_ι is not a boundary wavelet. Suppose without loss of generality,

$$
n_Q = \vec n, \quad \Phi^{j_\kappa}(t) = R + e^t \vec v, \quad \Phi^{j_\iota}(t) = R + e^t \vec w. \tag{3.17}
$$

If $j_\kappa = j_\iota$ and $\vec v = \vec w$, then $n_Q \cdot (P - Q) = 0$ and $k(P,Q)$ vanishes. For $\vec v \ne \vec w$, we get

$$
k(\Phi^{j_\kappa}(t), \Phi^{j_\iota}(s))|D\Phi^{j_\iota}(s)| = \frac{\vec n \cdot (e^s \vec w - e^t \vec v)}{|e^s \vec w - e^t \vec v|^2} e^s |\vec w| = \frac{\vec n \cdot \vec w - e^{t-s} \vec n \cdot \vec v}{|\vec w - e^{t-s}\vec v|^2} |\vec w|. \tag{3.18}
$$

This kernel function is smooth. Moreover, it is easy to see that any derivative of this function with respect to t or s can be estimated by a constant. Using the representation $f(\hat P_\kappa) = \sum_{i=1}^3 \alpha_i f(P_{\kappa,i})$ and the notation $P_{\kappa,i} = \Phi^{j_\kappa}(t_{\kappa,i})$, we conclude

$$
(A\psi_\iota)(\hat P_\kappa) = \sum_{i=1}^3 \alpha_i \int_{-\infty}^0 k(\Phi^{j_\kappa}(t_{\kappa,i}), \Phi^{j_\iota}(s))|D\Phi^{j_\iota}(s)| \psi^{l_\iota}_{k_\iota}(s) ds. \tag{3.19}
$$

Let Tay_1 stand for the Taylor series expansion up to linear terms of $k(\Phi^{j_\kappa}(t), \Phi^{j_\iota}(s))$ $|D\Phi^{j_\iota}(s)|$ with respect to t at the point $t = t_{\kappa,1}$. Furthermore, let Tay_2 stand for the Taylor series expansion up to linear terms of $k(\Phi^{j_\kappa}(t), \Phi^{j_\iota}(s))|D\Phi^{j_\iota}(s)| - Tay_1$ with respect to s at the midpoint $s = s_\iota$ of the support of ψ_ι. Then we set $Tay = Tay_1 + Tay_2$ and get

$$\left| k(\Phi^{j_\kappa}(t), \Phi^{j_\iota}(s)) |D\Phi^{j_\iota}(s)| - Tay \right| \le C|t - t_{\kappa,1}|^2 |s - s_\iota|^2 . \tag{3.20}$$

In view of the moment conditions of our wavelets, we know that $\hat{P}_\kappa$ vanishes at the linear function Tay_1 if $l_\kappa > 0$ and that $\psi^{l_\iota}_{k_\iota}$ is orthogonal to Tay_2 if $l_\iota > 0$ and if $\psi^{l_\iota}_{k_\iota}$ is not a boundary wavelet. Thus let us suppose $l_\kappa > 0$, $l_\iota > 0$ and that ψ_ι is not a boundary wavelet. From (3.20) we conclude

$$\begin{aligned} (A\psi_\iota)(\hat{P}_\kappa) &= \sum_{i=1}^{3} \alpha_i \int_{-\infty}^{0} \left\{ k(\Phi^{j_\kappa}(t_{\kappa,i}), \Phi^{j_\iota}(s)) |D\Phi^{j_\iota}(s)| - Tay \right\} \psi^{l_\iota}_{k_\iota}(s) ds, \\ |(A\psi_\iota)(\hat{P}_\kappa)| &\le C(diam\, supp\, \vartheta^{l_\kappa}_{k_\kappa})^2 (diam\, supp\, \psi^{l_\iota}_{k_\iota})^2 \int |\psi_\iota(Q)| d_\Gamma Q \qquad (3.21) \\ &\le C(h2^{lev - l_\kappa})^2 (h2^{lev - l_\iota})^2 \int |\psi_\iota(Q)| d_\Gamma Q. \end{aligned}$$

Since $\iota \in I^A \setminus I^A(\hat{P}_\kappa)$ implies $l_\kappa + l_\iota > lev$, we arrive at

$$\begin{aligned} \sum_{\iota \in I^A \setminus I^A(\hat{P}_\kappa)} |(A\psi_\iota)(\hat{P}_\kappa)| &\le Ch^4 2^{2\,lev} \sup_{Q\in\Gamma} \sum_{\iota \in I^A} |\psi_\iota(Q)| \qquad (3.22) \\ &\le Ch^4 [h^{-1} \log h^{-1}]^2 lev \ \le \ Ch^2 lev^3. \end{aligned}$$

Thus $\|B_N - B^c_N\|_{\mathcal{L}(l^\infty(I^A), l^\infty(I^T))} \le Ch^2 lev^3$. By Lemmas 3.4 and 3.3 as well as by $A_N = Tr^T_N B_N Tr^A_N$, we get assertion i).
Let us turn to the discretization error on the left-hand side of ii). We have to estimate the entries $C_{\kappa,\iota}$ of $C = B^c_N - B'_N$, where

$$\begin{aligned} C_{\kappa,\iota} &:= \sum_{i=1}^{3} \alpha_i \int_\Gamma k(P_{\kappa,i}, Q)[\psi_\iota(Q) - \psi_\iota(P_{\kappa,i})] d_Q\Gamma \qquad (3.23) \\ &\quad - \sum_{i=1}^{3} \alpha_i \sum_{\mu \in J} k(P_{\kappa,i}, Q_\mu)[\psi_\iota(Q_\mu) - \psi_\iota(P_{\kappa,i})]\omega_\mu =: \sum_{i=1}^{3} \alpha_i \{ Te^1_i + Te^2_i \psi_\iota(P_{\kappa,i}) \}, \\ Te^1_i &:= \int_\Gamma k(P_{\kappa,i}, Q)\psi_\iota(Q) d_Q\Gamma - \sum_{\mu\in J} k(P_{\kappa,i}, Q_\mu)\psi_\iota(Q_\mu)\omega_\mu, \\ Te^2_i &:= \int_\Gamma k(P_{\kappa,i}, Q) d_Q\Gamma - \sum_{\mu\in J} k(P_{\kappa,i}, Q_\mu)\omega_\mu. \end{aligned}$$

The quadrature error Te^2_i is the sum of the quadrature error taken over the subintervals adjacent to the corners and of that taken over the rest. Since the kernel (cf.(3.18)) is smooth over the rest, we get the usual $O(h^2_{qu})$ estimate for the trapezoidal rule. For the error over the subintervals adjacent to the corner points, we obtain the estimate (cf. the definition of $Q_1(f; 0, e^{-(N-1)h})$)

$$\int_{subinterval} |\partial_Q k(P_{\kappa,i}, Q)| d_Q\Gamma \cdot (\frac{m}{i_*}). \tag{3.24}$$

Here ∂_Q is the derivative in the tangential direction t_Q ($|t_Q| = 1$) of Γ and m is the length of the subinterval. Without loss of generality, we may suppose that the $P_{\kappa,i}$ and the subinterval are placed on two different sides of Γ adjacent to the corner point R. Hence,

$$\partial_Q \left\{ \frac{1}{2\pi} \frac{n\cdot(P-Q)}{|P-Q|^2} \right\} = \partial_Q \left\{ \frac{1}{2\pi} \frac{n\cdot(P-R)}{|P-Q|^2} \right\} = \frac{n\cdot(P-R)}{\pi|P-Q|^4} t_Q \cdot (Q-P), \quad (3.25)$$

$$|\partial_Q k(P_{\kappa,i},Q)| \leq 2\frac{|k(P_{\kappa,i},Q)|}{|P_{\kappa,i}-Q|}. \quad (3.26)$$

Since $P_{\kappa,i}$ is a collocation point, we get $|P_{\kappa,i}-Q| \geq C^{-1}\cdot m$ and (3.24) can be estimated by $C\int |k(P_{\kappa i},Q)|d_Q\Gamma/i_* \leq C/i_*$. Collecting terms, we arrive at

$$|Te_i^2| \leq C\left\{h_{qu}^2 + \frac{1}{i_*}\right\}. \quad (3.27)$$

The estimation of the quadrature error Te_i^1 over the subintervals adjacent to the corner points is analogous to that of Te_i^2 since the trial functions are constant over these subintervals. Thus let us suppose that $supp\,\psi_\iota$ does not contain a corner and apply the substitutions (3.17). We get

$$Te_i^1 = \int k(P_{\kappa,i},\Phi^{j_\iota}(s))|D\Phi^{j_\iota}(s)|\psi_{k_\iota}^{l_\iota}(s)ds - \sum_\lambda k(P_{\kappa,i},\Phi^{j_\iota}(s_\lambda))|D\Phi^{j_\iota}(s_\lambda)|\psi_{k_\iota}^{l_\iota}(s_\lambda)\hat{\omega}_\lambda, \quad (3.28)$$

where $\hat{\omega}_\lambda := e^{-s_\lambda}\tilde{\omega}_\lambda$ (cf. Sect.1.2). Now observe that our quadrature rule is exact at functions from the trial space. Hence, if we choose an $s' \in supp\,\psi_{k_\iota}^{l_\iota}$, we arrive at

$$\begin{aligned} Te_i^1 &= \int \left[k(P_{\kappa,i},\Phi^{j_\iota}(s))|D\Phi^{j_\iota}(s)| - k(P_{\kappa,i},\Phi^{j_\iota}(s'))|D\Phi^{j_\iota}(s')|\right]\psi_{k_\iota}^{l_\iota}(s)ds - \quad (3.29)\\ &\sum_\lambda \left[k(P_{\kappa,i},\Phi^{j_\iota}(s_\lambda))|D\Phi^{j_\iota}(s)| - k(P_{\kappa,i},\Phi^{j_\iota}(s')|D\Phi^{j_\iota}(s')|\right]\psi_{k_\iota}^{l_\iota}(s_\lambda)\hat{\omega}_\lambda. \end{aligned}$$

Taking into account the properties of the kernel $k(\Phi^{j_\kappa}(t),\Phi^{j_\iota}(s))|D\Phi^{j_\iota}(s)|$, it is not hard to derive

$$|Te_i^1| \leq C(diam\,supp\,\psi_{k_\iota}^{l_\iota})\cdot\int_\Gamma |\psi_\iota(Q)|d_Q\Gamma. \quad (3.30)$$

Taking into account that $diam\,supp\,\psi_{k_\iota}^{l_\iota} \sim h2^{lev-l_\iota}$ and that $h2^{lev} \sim lev$, we get the bound $C\cdot lev^{-3}\int|\psi_\iota(Q)|d_Q\Gamma$ for $|Te_i^1|$ if $l_\iota \geq 4[\log lev/\log 2]$. On the other hand, the usual first order estimate for the quadrature together with the smoothness of the kernel implies

$$|Te_i^1| \leq C h_{qu}\, sup_Q|\partial_Q\psi_\iota(Q)|\int_{supp\,\psi_\iota} d_Q\Gamma, \leq Ch_{qu}(h2^{lev-l_\iota})^{-1}\int_{supp\,\psi_\iota} d_Q\Gamma. \quad (3.31)$$

Together with $h_{qu} \leq h2^{lev-lev_0}$ and $lev_0 = 7[\log lev/\log 2]$ we conclude $|Te_i^1| \leq C\cdot lev^{-3}\int_{supp\,\psi_\iota} d_Q\Gamma$ if $l_\iota \leq 4[\log lev/\log 2]$. Hence, for any l_ι, we get the bound $C\cdot lev^{-3}\int_{supp\,\psi_\iota} d_Q\Gamma$.
From Equs. (3.23), (3.27), and the last estimation we conclude

$$\begin{aligned} |C_{\kappa,\iota}| &\leq C\left\{h_{qu}^2 + \frac{1}{i_*}\right\}\sum_{i=1}^3 |\psi_\iota(P_{\kappa,i})| + C\left\{\frac{1}{i_*} + lev^{-3}\right\}\sum_{i=1}^3 \int_{supp\,\psi_\iota} d_Q\Gamma, \quad (3.32)\\ \sum_{\iota\in I^A(\hat{P}_\kappa)} |C_{\kappa,\iota}| &\leq C\left\{h_{qu}^2 + \frac{1}{i_*}\right\} lev + C\left\{\frac{1}{i_*} + lev^{-3}\right\} lev, \end{aligned}$$

$$\leq \; C\left\{h_{qu}^2 + \frac{1}{i_*} + lev^{-3}\right\} lev.$$

In view of $h_{qu} \leq h2^{lev-lev_0}$, $\; i_* = lev^3$, and $lev_0 = 7[\log lev/\log 2]$, we conclude

$$\sum_{\iota\in I^A(\hat{P}_\kappa)} |C_{\kappa,\iota}| \leq C\left\{lev^2 2^{-2lev_0} + lev^{-3}\right\} lev \leq C/lev^2. \tag{3.33}$$

Hence, $\|B_N^c - B_N'\| \leq C/lev^2$ and Lemmas 3.4 and 3.3 lead to assertion ii). ∎

REMARK 3.7 *We conjecture that Theorem 3.6 holds also for the piecewise cubic wavelet algorithm of Sect.1.5. Of course, the second order convergence is to be replaced by fourth order convergence and the exponents of the logarithm change. The proof should be analogous to that presented above. The only open problem is to prove analogues of Lemmas 3.4 and 3.5. We have not tried to show the boundedness of the transforms corresponding to the boundary modification of the wavelets.*

4 ASYMPTOTIC RATES OF CONVERGENCE

Let x denote the solution of $Ax = (I + 2W)x = y$ and suppose the right-hand side y is continuous on Γ and infinitely differentiable on each closed side of Γ. Then the function x is infinitely differentiable at any point of Γ which is not a corner point. If R is a corner, then the asymptotics of $x(P)$ for $P \longrightarrow R$ takes the form $x(P) - x(R) \sim |P-R|^{\kappa_R}$, where $\kappa_R := \pi/\max(\alpha, 2\pi - \alpha)$ and α is the interior angle of the polygon Γ at R (cf. [37, 28]). In particular, x belongs to the Hölder class over Γ with exponent $\kappa_\Gamma := min\{\kappa_R,\ R \text{ corner of } \Gamma\}$ and the functions $(-\infty, 0] \ni s \mapsto x(\Phi^j(s))$, $j = 1, \ldots, K$ are smooth.

THEOREM 4.1 *Let x denote the exact solution of the double layer potential equation $Ax = y$ and suppose x_N is the approximate solution obtained by the algorithm of Sect.1.4, i.e., $x_N = \sum_{\iota\in I} \xi_\iota \varphi_\iota$ is obtained by solving $A_N^w\{\xi_\iota\}_{\iota\in I} = \{y(P_\kappa)\}_{\kappa\in I}$ iteratively. Note that $x_N = \sum_{\iota\in I^A} \eta_\iota \psi_\iota$ with the solution $\{\eta_\iota\}_{\iota\in I^A}$ from the equation $B_N'\{\eta_\iota\}_{\iota\in I^A} = \{y(\hat{P}_\kappa)\}_{\kappa\in I^T}$. If the right-hand side y is continuous on Γ and infinitely differentiable on each closed side of Γ, then there holds*

$$\|x - x_N\|_{L^\infty} \leq C\ \max\left(lev\, N^{-\zeta\kappa_\Gamma}, lev^4 h^2\right), \tag{4.1}$$

where C is independent of N and h.

COROLLARY 4.2 *The estimate (4.1) expressed in terms of the step size h or in terms of the degree of freedom N takes the form*

$$\|x - x_N\|_{L^\infty} \leq \begin{cases} C\ h^{\min(\zeta\kappa_\Gamma,2)}[\log h^{-1}]^{\mu_1} \\ C\ N^{-\min(\zeta\kappa_\Gamma,2)}[\log N]^{\mu_2}, \end{cases} \tag{4.2}$$

where $\mu_1 = -\zeta\kappa_\Gamma + 1$, $\mu_2 = 1$ for $\zeta\kappa_\Gamma < 2$ and $\mu_1 = 4$, $\mu_2 = 6$ for $\zeta\kappa_\Gamma \geq 2$.

PROOF: Let L_N denote the interpolation projection $L_N f := \sum_{\iota\in I} f(P_\iota)\varphi_\iota$. Then we can identify the function z_N of the trial space $im\, L_N$ with the sequence $\{z_N(P_\iota)\}_{\iota\in I} \in l^\infty(I)$ of its coefficients. Using $\|L_N\| = 1$ and the stability of $A_N^w = Tr_N^T B_N' Tr_N^A \in \mathcal{L}(im\, L_N)$

(cf. Theorem 3.6), we get

$$\begin{aligned} x - x_N &= x - L_N x + (A_N^w)^{-1} A_N^w L_N x - (A_N^w)^{-1} L_N y, \\ \|x - x_N\| &\leq \|x - L_N x\| + \|(A_N^w)^{-1}\| \|A_N^w L_N x - L_N A x\| \\ &\leq C\,\|x - L_N x\| + C\,\|A_N^w L_N x - L_N A L_N x\|. \end{aligned} \tag{4.3}$$

The operator $L_N A|_{im\, L_N}$ is nothing else than the collocation operator A_N of Sect.1.1. Hence, we obtain

$$\begin{aligned} \|x - x_N\| \;\leq\; & C\,\|x - L_N x\| + C\,\|Tr_N^T B_N^c Tr_N^A - A_N\|\,\|x\| + \\ & \|Tr_N^T\| \|[B_N^c - B_N']Tr_N^A L_N x\|. \end{aligned} \tag{4.4}$$

Using the smoothness of x and the special choice of the grids (cf. Sect.1.1), it is not hard to obtain

$$\|x - L_N x\|_{L^\infty} \leq C\ \max\left(h^2, N^{-\zeta\kappa_\Gamma}\right). \tag{4.5}$$

The term $\|Tr_N^T B_N^c Tr_N^A - A_N\|$ has been estimated in Theorem 3.6,i) and $\|Tr_N^T\|$ in Lemma 3.4. Hence, it remains to consider

$$\|[B_N^c - B_N']Tr_N^A L_N x\| = \sup_{\kappa \in I^T} |B_N' Tr_N^A L_N x(\hat{P}_\kappa) - B_N^c Tr_N^A L_N x(\hat{P}_\kappa)|. \tag{4.6}$$

Obviously, we have

$$B_N' Tr_N^A L_N x(\hat{P}_\kappa) - B_N^c Tr_N^A L_N x(\hat{P}_\kappa) = \sum_{i=1}^3 \alpha_i Te_i, \tag{4.7}$$

$$Te_i := \int k(P_{\kappa,i}, Q)[x_N^*(Q) - x_N^*(P_{\kappa,i})] d_Q\Gamma - \sum_\mu k(P_{\kappa,i}, Q_\mu)[x_N^*(Q_\mu) - x_N^*(P_{\kappa,i})]\omega_\mu,$$

where x_N^* is the compression $\sum_{\iota \in I^A(\hat{P}_\kappa)} \eta_\iota \psi_\iota$ of $L_N x = \sum_{\iota \in I^A} \eta_\iota \psi_\iota$.
Let us first estimate the quadrature error (4.7) over a subinterval Γ_a adjacent to the corner points. Without loss of generality we may suppose that $\hat{P}_\kappa$ is not a corner point and that $\hat{P}_\kappa$ and Γ_a belong to two different sides of Γ having the corner point R in common. For $Q \in \Gamma_a$, the value $x_N^*(Q)$ is equal to $L_N x(Q)$ and to the value of x at the corner point R. Moreover, $x_N^*(P_{\kappa,i}) = L_N x(P_{\kappa,i}) = x(P_{\kappa,i})$. Consequently, we get

$$|x_N^*(Q) - x_N^*(P_{\kappa,i})| = |x(R) - x(P_{\kappa,i})| \leq C|R - P_{\kappa,i}|^{\kappa_\Gamma}. \tag{4.8}$$

$$\begin{aligned} |\int_{\Gamma_a} k(P_{\kappa,i}, Q)[x_N^*(Q) - x_N^*(P_{\kappa,i})] d_Q\Gamma| &\leq C \int_{\Gamma_a} |P_{\kappa,i} - Q|^{-1} d_Q\Gamma\, |R - P_{\kappa,i}|^{\kappa_\Gamma} \quad (4.9) \\ &\leq C|R - P_{\kappa,i}|^{\kappa_\Gamma - 1} \int_{\Gamma_a} d_Q\Gamma \\ &\leq C|R - P_{\kappa,i}|^{\kappa_\Gamma - 1} N^{-\zeta} \leq N^{-\zeta\kappa_\Gamma}. \end{aligned}$$

The quadratures over Γ_a can be estimated similarly.
Now let us turn to the quadrature error over the union of all subintervals which are not adjacent to corner points. We write

$$\sum_{i=1}^3 \alpha_i Te_i \;=\; Te' + Te", \tag{4.10}$$

$$Te' = \int \left[\sum_{i=1}^{3} \alpha_i k(P_{\kappa,i}, Q) \right] x_N^*(Q) d_Q \Gamma - \sum_{\mu \in J} \left[\sum_{i=1}^{3} \alpha_i k(P_{\kappa,i}, Q_\mu) \right] x_N^*(Q_\mu) \omega_\mu,$$

$$Te" = \int \left[\sum_{i=1}^{3} \alpha_i x_N^*(P_{\kappa,i}) k(P_{\kappa,i}, Q) \right] d_Q \Gamma - \sum_{\mu \in J} \left[\sum_{i=1}^{3} \alpha_i x_N^*(P_{\kappa,i}) k(P_{\kappa,i}, Q_\mu) \right] \omega_\mu.$$

Without loss of generality we suppose that $\hat{P}_\kappa$ is not a corner point and that the domain of integration and $\hat{P}_\kappa$ belong to two different sides of Γ adjacent to a corner point R. Let the domain of integration be part of Γ_j. Using the substitutions (3.17), the quadrature error of Te' can be estimated by

$$h_{qu}^2 \cdot \int \left| \partial_s^2 \left\{ \left[\sum_{i=1}^{3} \alpha_i k(P_{\kappa,i}, \Phi^j(s)) |D\Phi^j(s)| \right] x_N^*(\Phi^j(s)) \right\} \right| ds. \tag{4.11}$$

Since the second derivative of this piecewise linear function is zero (Note that the points of discontinuity of the first derivative of the piecewise linear functions are node points of the trapezoidal rule.), we get

$$\partial_s^2 \left\{ \left[\sum_{i=1}^{3} \alpha_i k(P_{\kappa,i}, \Phi^j(s)) |D\Phi^j(s)| \right] x_N^*(\Phi^j(s)) \right\} =$$
$$\sum_{l=0}^{1} \begin{pmatrix} 2 \\ l \end{pmatrix} \partial_s^{2-l} \left[\sum_{i=1}^{3} \alpha_i k(P_{\kappa,i}, \Phi^j(s)) |D\Phi^j(s)| \right] \partial_s^l \left[x_N^*(\Phi^j(s)) \right]. \tag{4.12}$$

Now we take into account the smoothness of the kernel function (cf. the proof of Theorem 3.6) and apply the estimate (cf. Sect.1.3)

$$\begin{aligned} \left| \sum_{i=1}^{3} \alpha_i f(P_{\kappa,i}) \right| &= \left| f(\Phi^{j_\kappa}(t_{k_\kappa}^{l_\kappa})) - \frac{1}{2} \left\{ f(\Phi^{j_\kappa}(t_{k_\kappa,1}^{l_\kappa})) + f(\Phi^{j_\kappa}(t_{k_\kappa,2}^{l_\kappa})) \right\} \right| \\ &\leq C \sup_{t_{k_\kappa,1}^{l_\kappa} \leq t \leq t_{k_\kappa,2}^{l_\kappa}} \left| \partial_t^2 \left[f \circ \Phi^{j_\kappa} \right](t) \right| |t_{k_\kappa,1}^{l_\kappa} - t_{k_\kappa,2}^{l_\kappa}|^2, \\ &\leq C \sup_{t \in supp\, \vartheta_{k_\kappa}^{l_\kappa}} \left| \partial_t^2 \left[f \circ \Phi^{j_\kappa} \right](t) \right| [diam\, supp\, \vartheta_{k_\kappa}^{l_\kappa}]^2 \end{aligned} \tag{4.13}$$

to get

$$\int \left| \partial_s^2 \left\{ \left[\sum_{i=1}^{3} \alpha_i k(P_{\kappa,i}, \Phi^j(s)) |D\Phi^j(s)| \right] x_N^*(\Phi^j(s)) \right\} \right| ds \leq$$
$$C [diam\, supp\, \vartheta_{k_\kappa}^{l_\kappa}]^2 \left\{ \sup |[x_N^* \circ \Phi^j](s)| + \sup |\partial_s [x_N^* \circ \Phi^j](s)| \right\}. \tag{4.14}$$

Similarly we can estimate $Te"$ by

$$h_{qu}^2 \cdot \int \left| \partial_s^2 \left[\sum_{i=1}^{3} \alpha_i x_N^*(P_{\kappa,i}) k(P_{\kappa,i}, \Phi^j(s)) |D\Phi^j(s)| \right] \right| ds = \tag{4.15}$$
$$h_{qu}^2 \cdot \int \left| \partial_s^2 \left[\sum_{i=1}^{3} \alpha_i x(P_{\kappa,i}) k(P_{\kappa,i}, \Phi^j(s)) |D\Phi^j(s)| \right] \right| ds \leq$$
$$C \cdot h_{qu}^2 \left\{ \sup |[x \circ \Phi^j](s)| + \sup |\partial_s [x \circ \Phi^j](s)| + \sup |\partial_s^2 [x \circ \Phi^j](s)| \right\} \cdot [diam\, supp\, \vartheta_{k_\kappa}^{l_\kappa}]^2$$

$$\leq C \cdot h_{qu}^2 [diam\, supp\, \vartheta_{k_\kappa}^{l_\kappa}]^2.$$

Now it remains to estimate $\sup |[x_N^* \circ \Phi^j](s)|$ and $\sup |\partial_s [x_N^* \circ \Phi^j](s)|$. Since the estimate of $\sup |x_N^* \circ \Phi^j|$ is similar to that of $\sup |\partial_s [x_N^* \circ \Phi^j](s)|$, we shall concentrate on the latter. From

$$\partial_s \left[\sum_{l,k} \eta_k^l \psi(s/(h2^{lev-l}) - k) \right] = \sum_{l,k} \eta_k^l / (h2^{lev-l}) \psi'(s/(h2^{lev-l}) - k) \tag{4.16}$$

and from the boundedness of $supp\, \psi = supp\, \psi'$ (cf. (1.15)), it is easy to see that

$$\begin{aligned} \sup |\partial_s [x_N^* \circ \Phi^j](s)| &\leq C \cdot lev\ sup_{\iota \in I^A(\hat{P}_\kappa)} \left| \eta_\iota / (h2^{lev - l_\iota}) \right| \\ &\leq C \cdot lev\ sup_{\iota \in I} \left| \eta_\iota / (h2^{lev - l_\iota}) \right|. \end{aligned} \tag{4.17}$$

In order to estimate $\eta_\iota / (h2^{lev-l_\iota})$, we introduce the extension $\tilde{z}_N$ of $[-(N-1)h, 0] \ni s \mapsto L_N x(\Phi^j(s))$ by

$$\tilde{z}_N(s) := \begin{cases} \dots & \\ L_N x(\Phi^j(s - 2(N-1)h)) & \text{if } (N-1)h \leq s \leq 2(N-1)h \\ L_N x(\Phi^j(-s)) & \text{if } 0 \leq s \leq (N-1)h \\ L_N x(\Phi^j(s)) & \text{if } -(N-1)h \leq s \leq 0 \\ L_N x(\Phi^j(-2(N-1)h - s)) & \text{if } -2(N-1)h \leq s \leq -(N-1)h \\ \dots & . \end{cases} \tag{4.18}$$

Note that, since the wavelets over $[-\infty, 0]$ are defined with the help of reflection, the wavelet coefficients of $L_N x$ corresponding to $\{\psi_k^l\}$ are the same as those of $\tilde{z}_N$ corresponding to the wavelet basis defined by (1.16) over the real axis. The wavelet coefficients of $\tilde{z}_N$ can be computed via the biorthogonal wavelet functions $\psi_{k,l}^d$ (cf. the proof of Lemma 3.5) such that

$$\tilde{z}_N = \sum_{l=0}^{lev} \sum_{k \in \mathbb{Z}} (\psi_{l,k}^d, \tilde{z}_N) \tilde{\psi}_k^l. \tag{4.19}$$

Now fix l with $0 < l \leq lev$. Surely, $\tilde{z}_N$ has a bounded derivative. Consequently, there exists a piecewise linear function $\tilde{z}_N^1$ in $span\, \{\tilde{\psi}_k^m,\ 0 \leq m < l,\ k \in \mathbb{Z}\}$ such that $\|\tilde{z}_N - \tilde{z}_N^1\|_{L^\infty} \leq C(h2^{lev-l+1})$. Since $\psi_{l,k}^d$ is orthogonal to $\tilde{z}_N^1$, we get

$$\begin{aligned} \eta_k^l &= (\psi_{l,k}^d, \tilde{z}_N) = (\psi_{l,k}^d, \tilde{z}_N - \tilde{z}_N^1), \\ |\eta_k^l| &\leq \|\psi_{l,k}^d\|_{L^1} \|\tilde{z}_N - \tilde{z}_N^1\|_{L^\infty} \leq C h 2^{lev-l}. \end{aligned} \tag{4.20}$$

If $l = 0$, then $|\eta_k^0| = (\psi_{0,k}^d, \tilde{z}_N) \leq C \leq C(h2^{lev})$.

Collecting the estimates (4.10)-(4.20), we conclude that the quadrature error $|B_N' Tr_N^A L_N x(\tilde{P}_\kappa) - B_N^c Tr_N^A L_N x(\tilde{P}_\kappa)|$ taken over all the subintervals of Γ which are not adjacent to a corner can be estimated by

$$\begin{aligned} C h_{qu}^2 [diam\, supp\, \vartheta_{k_\kappa}^{l_\kappa}]^2 \cdot lev &\leq C (h2^{l_\kappa})^2 (h2^{lev-l_\kappa})^2 \cdot lev \\ &\leq C \cdot lev\, (h^2 2^{lev})^2 \leq C \cdot lev^3 h^2. \end{aligned} \tag{4.21}$$

From this estimate, Lemma 3.4, Theorem 3.6, and the inequalities (4.4)-(4.6), and (4.9) we obtain

$$\|x - x_N\| \leq C \left\{ \max(h^2, N^{-\zeta\kappa_\Gamma}) + h^2 lev^4 + lev[N^{-\zeta\kappa_\Gamma} + lev^3 h^2] \right\} \tag{4.22}$$

which proves (4.1). ∎

REMARK 4.3 *If the conjecture of Remark 3.7 is true, then a result analogous to Theorem 4.1 can be proved for the piecewise cubic collocation together with the wavelet algorithm of Sect.1.5. Clearly, the exponent two in (4.1) is to be replaced by four since this exponent corresponds to the polynomial degree of the trial functions and to the convergence order of the quadrature rule.*

REMARK 4.4 *Let us note that a better compression than that of Sect.1.4 is possible. Indeed, define $I^A(\hat{P}_\kappa)$ to be the set of all $\iota \in I^A$ such that $\psi_\iota(P_{\kappa,i}) \neq 0,\ i = 1,2,3$ or that ψ_ι is a boundary wavelet or that $l_\iota \leq [lev - l_\kappa]/2$. In this case the bound of Theorem 3.6,i) takes the form $C \cdot h \cdot lev^5$. However, arguing analogously to (4.20), one can prove that $|\eta_\iota| \leq C \cdot (h 2^{lev - l_\iota})^2$ for $L_N x = \sum_{\iota \in I^A} \eta_\iota \psi_\iota$. Using this, it is not hard to get (cf. (3.21))*

$$\begin{aligned} \|[Tr_N^T B_N^c Tr_N^A - A_N] L_N x\| &\leq \|Tr_N^T\| \cdot \sup_{\kappa \in I^T} \sum_{\iota \in I^A \setminus I^A(\hat{P}_\kappa)} |(A\psi_\iota)(\hat{P}_\kappa)\, \eta_\iota| \\ &\leq C \cdot h^2 lev^5. \end{aligned} \tag{4.23}$$

Hence, we arrive at a convergence estimate of $\|x - x_N\| \leq C \cdot h^2 lev^5$ for this kind of compression if $\zeta\kappa_\Gamma > 2$.

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REFERENCES

1. B.K. Alpert, *Construction of simple multiscale bases for fast matrix operation*, in *Wavelets and their applications*, M.B. Ruskai, G. Beylkin, R. Coifman, I. Daubechies, S. Mallat, Y. Meyer, and L. Raphael, eds: Jones and Bartlett Publishers, Boston, 211–226, 1992.

2. B.K. Alpert, G. Beylkin, R. Coifman, and V. Rokhlin, *Wavelet-like bases for the fast solution of second-kind integral equations*, SIAM J. Sci. Statist. Comput. **14** (1990), 159–184.

3. S. Amini and I.H. Sloan, *Collocation methods for second kind integral equations with non-compact operators*, J. Integral Equations Appl. **2** (1989), 1–30.

4. B.A. Amosov, *On the approximate solution of elliptic pseudodifferential equations over smooth and closed curves* (in russian), Zeitschr. Anal. Anwendungen **9** (1990), 545–563.

5. L. Andersson, N. Hall, B. Jawerth, and G. Peters, *Wavelets on closed subsets of the real line*, in *Topics in the Theory and Applications of Wavelets*, L.L. Schumaker and G. Webbs, eds., Academic Press, Boston, 1–61, 1993.

6. K.E. Atkinson and I.G. Graham, *Iterative solution of linear systems arising from the boundary integral method*, SIAM J. Sci. Statist. Comput. **13** (1992), 694–722.

7. D. Berthold, W. Hoppe, and B. Silbermann, *A fast algorithm for solving the generalized airfoil equation*, J. Comput. Appl. Math. **43** (1992), 185–219.

8. G. Beylkin, R. Coifman, and V. Rokhlin, *Fast wavelet transforms and numerical algorithms I*, Comm. Pure Appl. Math. **44** (1991), 141–183.

9. A. Böttcher and B. Silbermann, *Analysis of Toeplitz operators*, Akademie-Verlag, Berlin, 1989.

10. A. Brandt and A.A. Lubrecht, *Multilevel matrix multiplication and fast solution of integral equation*, J. Comput. Phys. **90** (1991), 348–370.

11. G. Bruhn and W.L. Wendland, *Über die näherungsweise Lösung von linearen Funktionalgleichungen*, in *Funktionalanalysis, Approximationstheorie, Numerische Mathematik*, L. Collatz, G. Meinardus, and H. Unger, eds., ISNM Vol.7, Birkhäuser, Basel, 136–164, 1967.

12. F.X. Canning, *Sparse representation for solving integral equations with oscillatory kernels*, SIAM J. Sci. Statist. Comp. **13** (1992), 71–87.

13. G. Chandler and I.G. Graham, *High order methods for linear functionals of solutions of second kind integral equations*, SIAM J. Numer. Anal. **25** (1988), 1118–1137.

14. C.K. Chui, *An introduction to wavelets*, Academic Press, Boston, 1992.

15. C.K. Chui and E. Quak, *Wavelets on a bounded interval*, in *Numerical methods of Approximation theory*, D. Braess and L.L. Schumaker, eds., Birkhäuser Verlag, Basel, 53–75, 1992.

16. A. Cohen, I. Daubechies, and J.-C. Feauveau, *Biorthogonal bases of compactly supported wavelets*, Comm. Pure and Appl. Math. **45** (1992), 485–560.

17. A. Cohen, I. Daubechies, and P. Vial, *Wavelets on the interval and fast wavelet transforms*, Applied and Computational Harmonic Analysis **1** (1993), 54–81.

18. I. Daubechies, *Ten lectures on wavelets*, CBMS Lecture Notes, No. 61, SIAM, Philadelphia, 1992.

19. W. Dahmen, S. Prößdorf, and R. Schneider, *Multiscale methods for pseudo-differential equations*, in *Recent Advances in Wavelet Analysis*, L.L. Schumaker and G. Webb, eds., Wavelet Analysis and its Application **3**, (1994), 191–235 .

20. W. Dahmen, S. Prößdorf, and R. Schneider, *Wavelet approximation methods for pseudo-differential equations I: Stability and convergence*, Math. Zeitschr. **215** (1994), 583–620.

21. W. Dahmen, S. Prößdorf, and R. Schneider, *Wavelet approximation methods for pseudo-differential equations II: Matrix compression and fast solution*, Advances in Comp. Math. **1** (1993), 259-335.

22. W. Dahmen, B. Kleemann, S. Prößdorf, and R. Schneider, *A multiscale method for the double layer potential equation on a polyhedron*, in *Advances of Computational Mathematics*, H.P. Dikshit and C.A. Micchelli, eds., World Scientific Publ. Co., Inc., New Delhi, 15–57, 1994.

23. W. Dahmen, S. Prößdorf, and R. Schneider, *Multiscale methods for pseudo-differential equations on smooth manifolds*, in *Wavelets: Theory, Algorithms, and Applications*, C.K. Chui, L. Montefuaco, and L. Puccio., eds., Academic Press, 1–40, 1994.

24. M. Dorobantu, *Potential integral equations of the 2D Laplace operator in wavelet basis*, Preprint TRITA-NA-9401, Royal Institute of Technology, University of Stockholm, 1994.

25. D. Elliott and S. Prößdorf, *An algorithm for the approximate solution of integral equations of Mellin type*, Numer. Math., to appear.

26. J. Elschner, *On spline approximation for a class of non–compact integral equations*, Math. Nachr. **146** (1990), 271–321.

27. J. Elschner, *The h-p-version of spline approximation methods for Mellin convolution equations*, J. Integral Equations Appl. **5** (1993), 47–73.

28. J. Elschner, *Asymptotics of solutions to pseudodifferential equations of Mellin type*, Math. Nachr. **130** (1987), 267–305.

29. I.G. Graham, L. Qun, and X. Rue-feng, *Extrapolation of Nyström solutions of boundary integral equations on non-smooth domains*, J. Comput. Math. **10** (1992), 231–244.

30. L. Greengard and V. Rokhlin, *A fast algorithm for particle simulation*, J. Comput. Phys. **73** (1987), 325–348.

31. I. Gohberg and N. Krupnik, *One-dimensional linear singular integral equations*, Vols.I and II , *Operator Theory: Advances and Applications*, Vols.53 and 54, Birkhäuser Verlag, Basel, 1992, originally in russian, Shtiintsa, Kishinev, 1973.

32. W. Hackbusch and Z.P. Nowak, *On the fast matrix multiplication in the boundary element method by panel clustering*, Numer. Math. **54** (1989), 463–491.

33. A. Harten and I. Yad-Shalom, *Fast multiresolution algorithms for matrix-vector multiplication*, SIAM J. Numer. Anal. **31** (1994), 1191–1218.

34. Y. Jeon, *A Nyström method for boundary integral equations on domains with piecewise smooth boundary*, J. Integral Equations Appl. **5** (1993), 221–242.

35. A.V. Kozak, *A local principle in the theory of projection methods* (in russian), DAN SSSR **212** (1973), 1287–1289, translation from russian, Soviet Math. Doklady **14** (1974), 1580–1583.

36. R. Kreß, *A Nyström method for boundary integral equations in domains with corners*, Numer. Math. **58** (1990), 145–161.

37. V.G. Mazya, *Boundary integral equations*, in *Encyclopaedia of Math. Sciences, vol.*, vol. 27, *Analysis* I, V.G. Mazya and S.M. Nikol'skii, eds., Springer-Verlag, Berlin, Heidelberg, 127–222, 1991.

38. W. McLean, *Boundary integral methods for the Laplace equation*, Ph.D.Thesis, Canberra, 1985.

39. G. Mastroianni and G. Monegato, *Nyström interpolants based on the zeros of Legendre polynomials for a non-compact integral operator equation*, IMA J.Numer.Anal. **14** (1993), 81-95.

40. T.v. Petersdorff and C. Schwab, *Wavelet approximations for first kind boundary integral equations on polygons*, Technical Note BN-1157, Institute for Physical Science and Technology, University of Maryland at College Park, 1994.

41. S. Prößdorf, *Ein Lokalisierungsprinzip in der Theorie der Splineapproximationen und einige Anwendungen*, Math. Nachr. **119** (1984), 239–255.

42. S. Prößdorf and B. Silbermann, *Numerical analysis for integral and related operator equations*, Birkhäuser Verlag, Basel, 1991.

43. L. Qun and X. Rue-feng, *Extrapolation for the approximations to the solution of a boundary integral equation in polygonal domains*, J. Comput. Math. **7** (1989), 174–181.

44. A. Rathsfeld, *Eine Quadraturformelmethode für Mellin-Operatoren nullter Ordnung*, Math. Nachr. **137** (1988), 321–354.

45. A. Rathsfeld, *Iterative solution of linear systems arising from the double-layer potential equation over curves with corners*, Math. Meth. Appl. Sci. **16** (1993), 443–455.

46. A. Rathsfeld, *Nyström's method and iterative solvers for the solution of the double layer potential equation over polyhedral boundaries*, SIAM J. Numer. Anal. **32** (1995).

47. A. Rathsfeld, *Piecewise polynomial collocation for the double layer potential equation over polyhedral boundaries*, in *Boundary value problems and integral equations in non-smooth domains*, M. Costabel, M. Dauge, and S. Nicaise, eds., Lecture Notes in Pure and Applied Mathematics Series/167, Marcel Dekker, New York, Basel, 219–253, 1994.

48. V. Rokhlin, *Rapid solution of integral equations of classical potential theory*, J. Comput. Phys. **60** (1983), 187–207.

49. V. Rokhlin, *Rapid solution of integral equations of scattering theory in two dimensions*, J. Comput. Phys. **86** (1990), 414–439.

50. J.E. Romate, *On the use of conjugate gradient-type methods for boundary integral equations*, Comput. Mech. **12** (1993), 214–232.

51. J. Saranen and G. Vainikko, *Fast solvers of integral and pseudodifferential equations on closed curves*, to appear.

52. S. Sauters, *Der Aufwand der Panel-Clustering-Methode für Integralgleichungen*, Bericht Nr. 9115, Inst. f. Inform. u. Prakt. Math., Christian-Albrechts-Univ. Kiel, 1991.

53. H. Schippers, *Multi-grid methods for boundary integral equations*, Numer. Math. **46** (1985), 351–363.

54. B. Silbermann, *Lokale Theorie des Reduktionsverfahrens*, Math. Nachr. **104** (1981), 137–146.

55. G. Strang, *Wavelets and dilation equations: A brief introduction*, SIAM Review **31** (1989), 614–627.